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ACTIONS OF CARTAN SUBGROUPS

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ABSTRACT

Using affine Bruhat-Tits buildings, we associate certain subshifts of finite type to systems arising from the action of a Cartan subgroup of a p-adic semisimple Chevalley group on compact quotient spaces $\Gamma\backslash G$. These are used to study the resulting dynamical systems.

Many interesting dynamical systems may be obtained as follows: Let G be a topological group, $\Gamma < G$ a lattice and $H < G$ a subgroup. H acts on the space $\Gamma\backslash G$ by translations, T_h: $\Gamma\backslash G \to \Gamma\backslash G$ is given by $T_h(\Gamma g) = \Gamma gh$. This action preserves the normalized G-invariant measure μ on $\Gamma\backslash G$ and we obtain a system $(\Gamma\backslash G, \mathcal{B}, \mu, H)$ where $\mathcal{B}$ is the Borel σ-algebra of $\Gamma\backslash G$. A special class of such systems, which is the subject of this paper, are those obtained when G is a semisimple Chevalley group over $\mathbb{Q}_p$ and H is a subgroup of a Cartan subgroup of G. One of the main tools for studying dynamical systems with actions of $\mathbb{Z}$, i.e. systems of the form $(X, \mathcal{F}, \nu, T)$ where T: $X \to X$ is a measure preserving transformation, is constructing a "symbolic description" of the system. By this we mean the use of a partition of X into finitely many pieces in order to obtain a morphism of dynamical systems Φ: $(X, \mathcal{F}, \nu, T) \to (S^{\mathbb{Z}}, \mathcal{M}, \nu', \sigma)$, where S is a finite set indexing the partition and σ: $S^{\mathbb{Z}} \to S^{\mathbb{Z}}$ is the shift map. Obtaining such a map is especially useful when the image is a Markov shift, i.e. $\Phi(X) \subset S^{\mathbb{Z}}$

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is a subshift of finite type (a topological Markov shift) and the measure ν' is a Markov measure on it. For actions of groups other than $\mathbb{Z}$ such "(topological) Markov Partitions" are harder to obtain. Also the structure of higher dimensional subshifts of finite type ("tilings") is in general much more complicated than that of the one dimensional ones.

Semisimple Chevalley groups over $\mathbb{Q}_p$ are associated with affine Bruhat-Tits buildings. We will show how these buildings may be used to obtain subshifts of finite type which are factors of the systems we are interested in. In general these will be higher dimensional subshifts of finite type ("tiling systems"). These factors will be used to study the systems $(\Gamma\backslash G, \mathcal{B}, \mu, H)$. The main result concerning such groups that we obtain is the following:

THEOREM 3.1: *Let G be a semisimple Chevalley group over $\mathbb{Q}_p$, $\Gamma < G$ an irreducible torsion free lattice, $A < G$ a split Cartan subgroup of G, $\mathcal{B}$ the Borel σ-algebra of $\Gamma\backslash G$, μ Haar probability measure on $\Gamma\backslash G$, $T < A$ the maximal compact subgroup of A.*

(1) *Let $H < A$ be a closed subgroup, $d = \operatorname{rank} H$ (i.e., $H/H \cap T \cong \mathbb{Z}^d$). Then there exists a d-dimensional subshift of finite type $(\Omega, \mathcal{F}, \nu, H)$ on which H acts via $H/H\cap T$ so that $(\Gamma\backslash G, \mathcal{B}, \mu, H)$ is a compact affine extension of $(\Omega, \mathcal{F}, \nu, H)$. When H contains a regular element the extension is a compact group extension.*
(2) *The compact A orbits in $\Gamma\backslash G$ are dense.*
(3) *Let ${\mathbb{Q}_p}^* \cong H' < A$ be a regular one parameter subgroup, let $H = TH'$, η Haar measure on H normalized so that $\eta(T) = 1$. η induces a measure on compact H orbits in $\Gamma\backslash G$. The measure of a compact H orbit is a natural number. Denote by a_n the number of compact orbits of measure n. $a_n < \infty$, and*

$$\exp \sum_{n=1}^{\infty} \sum_{d|n} \frac{d a_d}{n} x^n = \frac{1}{\det(I - xM)}$$

where M is the adjacency matrix of a corresponding one-dimensional subshift of finite type.
(4) *(Notations as in (3).) Let μ_N be the probability measure obtained by normalizing the sum of the measures induced from η on the compact H orbits of measure $\leq N$. Then $\lim_{N\to\infty} \mu_N = \mu$ in the weak* topology.*

(5) *For any $g \in A$ such that $< g >$ is not compact, the (one-dimensional) system* $(\Gamma\backslash G, \mathcal{B}, \mu, T_g)$ *is Bernoulli.*

(6) *The directional entropy function* $h_\mu\colon A \to \mathbb{R}^+$ *is piecewise linear.*

It should be noted that the denseness of the compact A orbits for semisimple Lie groups over $\mathbb{R}$ was proved by G. G. Mostow [Mos], and by G. Prasad [Pr], via different methods, for the case of p-adic groups. The Bernoullicity of T_g for $g \in A \setminus T$ was proved using a different method by S. G. Dani [D]. The methods developed in this work are applied in [Moz2] to study closures of orbits of the maximal split Cartan subgroups for $G = \mathrm{PGL}(\mathbb{Q}_p) \times \mathrm{PGL}(\mathbb{Q}_l)$.

Section 1 contains a short summary of the definitions and basic properties of Bruhat-Tits buildings we will be using. Section 2 describes the construction of the subshift of finite type for systems as above where G is a group acting on an affine Bruhat-Tits building. Some special properties of these subshifts of finite type are proven and used to obtain information about the original system. In section 3 we apply the results of section 2 to systems arising from semisimple Chevalley groups over $\mathbb{Q}_p$, and give some applications and examples.

1. Bruhat-Tits buildings

We review here briefly the main properties of Bruhat-Tits buildings we will be using. We follow Brown's book [B], see also [B-T1], [B-T2], [Hi], [Ti1], [Ti2] and [Ti3].

Definitions:

(1) A finite dimensional simplicial complex is called a **chamber complex** if all the maximal cells — chambers — have the same dimension and any two chambers may be connected by a **gallery** — a sequence of chambers so that consecutive ones have a common codimension 1 face.

(2) A labelled chamber complex is a chamber complex together with a labelling of the vertices by a set of labels so that the vertices of any maximal simplex are in one to one correspondence with the set of labels.

(3) **Coxeter complex.** Let (W, S) be a Coxeter group. Associate with it a complex obtained from the partially ordered set whose elements are the cosets $w < S' >$ where $w \in W$, $S' \subset S$, ordered by $A \prec B$ if $B \subset A$. Any complex isomorphic to such a complex will be called a **Coxeter complex**. This is equivalent to the complex obtained by viewing (W, S)

as a reflection group of a Euclidean space and the complex is the complex obtained by the partition of the space by the reflection hyperplanes. A Coxeter complex is a **thin chamber complex** i.e. every codimension 1 cell belongs to exactly two chambers.

(4) An **affine Coxeter group** is a group of isometries of an affine space generated by reflections in affine hyperplanes belonging to an invariant set $\mathcal{H}$ of affine hyperplanes, s.t. every point in the space has an open set which intersects only finitely many hyperplanes from $\mathcal{H}$ and the geometrical realization of the corresponding combinatorial complex is an affine space. This complex is called an **affine Coxeter complex**.

(5) A **Bruhat-Tits building** (or just a **building**) is a complex Δ together with a collection of subcomplexes called **apartments** satisfying the following properties:

   (B0) Every apartment is a Coxeter complex.

   (B1) For each pair of cells $A, B \in \Delta$ there exists an apartment containing it.

   (B2) If Σ, Σ' are two apartments containing A and B, then there exists an isomorphism $\varphi: \Sigma \to \Sigma'$ which stabilizes A, B pointwise.

(6) A building Δ is called an **affine** building if each apartment is an affine Coxeter complex. We shall always assume it is a locally finite complex.

(7) Let G be a group of automorphisms of a building Δ. We will say that G acts **strongly transitively** if for any apartment Σ and a chamber $C \in \Sigma$ and an apartment Σ' and $C' \in \Sigma'$ there exists $g \in G$ so that $g\Sigma = \Sigma'$ and $gC = C'$.

(8) A map $\varphi: \mathcal{C} \to \mathcal{D}$ between two complexes will be called **admissible** if:

   (i) φ is a local homeomorphism.

   (ii) When $\mathcal{C}$ and $\mathcal{D}$ are labelled complexes, φ preserves the labelling.

(9) Let G be a group acting strongly transitively on a building Δ. Fix an apartment $\mathcal{A}$ and a chamber $C \in \mathcal{A}$. We define several subgroups of G:

$$B = \{g \in G \mid gC = C \text{ pointwise}\},$$

$$N = \{g \in G \mid g\mathcal{A} = \mathcal{A}\},$$

$$T = \{g \in N \mid g_{|\mathcal{A}} = \mathrm{id}\}.$$

When Δ is an affine building we will denote:

$$A = \{g \in N \mid g \text{ acts on } \mathcal{A} \text{ by translation}\}.$$

B is a **Borel** subgroup of G, $W = N/T$ the **Weyl group**, A is a **Cartan subgroup.**

(10) A **minimal gallery** connecting two chambers $C, D \in \Delta$ is a sequence of chambers $C_0 = C, C_1, \ldots, C_n = D$ s.t. the chambers C_i and C_{i+1} have a common face of codimension 1 and its length, n, is minimal.

(11) A subcomplex is called **convex** if it contains every minimal gallery connecting any two chambers of it.

(12) A building may have several apartment systems. An apartment system is **complete** if it contains any other apartment system. We will deal only with buildings with their (unique) complete apartment system.

THEOREM 1.1 (see [B] IV.5–6): *An affine building is contractible.*

Let $\mathcal{D}$ be a subset of an apartment $\mathcal{A}$ in an affine building. The convex hull of $\mathcal{D}$, $[\mathcal{D}]$, is the intersection of all the halfspaces of $\mathcal{A}$ containing $\mathcal{D}$ bounded by one of the hyperplanes partitioning $\mathcal{A}$ into chambers. This is an easy consequence of the fact that the chambers whose interior intersects a straight interval in an apartment form a minimal gallery.

THEOREM 1.2 (see [B] VI. Theorem 2): *A subcomplex of an affine building Δ which is either convex or has nonempty interior and is isometric to a part of an apartment (i.e. to a subset of a Euclidean space) is contained in an apartment.*

PROPOSITION 1.1: *Let $\mathcal{D} \subset \mathcal{A}$ be a convex subcomplex of maximal dimension. An admissible map $\varphi \colon \mathcal{D} \to \Delta$ is an isometry.*

Proof: It is enough to prove the assertion for $\mathcal{D}$ a finite complex. We will show, by induction, that if $\mathcal{L} = (C_0, C_1, \ldots, C_n)$ is a minimal gallery and $\varphi \colon \mathcal{L} \to \Delta$ an admissible map, then φ is an isometry and $\varphi(\mathcal{L})$ is contained in an apartment. Let $\mathcal{L}' = (C_0, C_1, \ldots, C_{n-1})$. By the induction hypotheses it follows that $\varphi_{|_{\mathcal{L}'}} \colon \mathcal{L}' \to \Delta$ is an isometry and $\varphi(\mathcal{L}')$ is contained in an apartment Σ'. Let $\rho \colon \Delta \to \Sigma'$ be the retraction with respect to the apartment Σ' and the chamber $\varphi(C_{n-1})$ (see [B] IV.3). The chamber $\rho(\varphi(C_n))$ is in Σ'. It is different from $\varphi(C_{n-1})$ and adjacent to it via the face $\varphi(C_{n-1} \cap C_n)$. Since $\varphi_{|_{\mathcal{L}'}}$ is an isometry, it follows that $\rho \circ \varphi \colon \mathcal{L} \to \Sigma'$ is an isometry. Since ρ and φ do not expand distances it follows that

$\varphi\colon \mathcal{L} \to \Delta$ is an isometry and $\varphi(\mathcal{L})$ is a minimal gallery connecting the chambers $\varphi(C_0)$ and $\varphi(C_n)$. Let Σ be any apartment containing $\varphi(C_0)$ and $\varphi(C_n)$. Since an apartment contains any minimal gallery connecting two chambers of it (see [B] IV.4) it follows that $\varphi(\mathcal{L}) \subset \Sigma$. ∎

PROPOSITION 1.2: *The stabilizer $G_x = \{g \in G \mid gx = x\}$ of a point $x \in \Delta$, Δ an affine building, is a compact subgroup (w.r.t. the product topology).*

Proof: Follows immediately from the fact that the building is a locally finite complex. ∎

2. Systems corresponding to groups acting on an affine building

Let Δ be an affine building with (the) complete apartments system, $r = \dim \Delta$. Let G be a group of automorphisms of Δ acting on it strongly transitively. Endow G with the topology induced from the product topology on Δ^{Δ}. Let $\Gamma < G$ be a uniform lattice. Assume that Γ is torsion free. Let A be the Cartan subgroup of G, $\mathcal{B}$ the Borel σ-algebra of $\Gamma\backslash G$, μ the G-invariant probability measure on $\Gamma\backslash G$. G acts on $\Gamma\backslash G$ by translations: $T_{g_0}(\Gamma g) = \Gamma g g_0$. In this section we will study systems of the form $(\Gamma\backslash G, \mathcal{B}, \mu, H)$ where H is a subgroup of A.

PROPOSITION 2.1: *Γ acts freely on Δ.*

Proof: This follows immediately from the fact that the stabilizer of a point in Δ is compact, together with the discreteness of Γ and the assumption that Γ is torsion free. ∎

Since Γ is a group of automorphisms of the complex Δ we have a quotient complex $Y = \Gamma\backslash\Delta$. Let $\pi\colon \Delta \to Y$ be the natural projection.

PROPOSITION 2.2: *Y is a finite complex. The labelling of Δ induces a labelling of Y. Δ is the universal covering space of Y with Γ the covering group and $\pi\colon \Delta \to Y$ a covering map.*

Proof: Fix a chamber C in Δ. For any chamber D in Y denote: $G(C, D) = \{g \in G \mid \pi(gC) = D\}$. $G(C, D)$ is an open set which is a union of right Γ-cosets. If $D, D' \in Y$, $D \neq D'$ then $G(C, D) \cap G(C, D') = \emptyset$. Since G acts transitively on Δ and π is surjective it follows that $G(C, D) \neq \emptyset$ for any $D \in Y$. Hence $\Gamma\backslash G = \bigcup_{D\in Y} G(C, D)$ is a disjoint open cover. Since $\Gamma\backslash G$ is compact, it is a finite cover. Hence Y is finite. The rest of the proposition follows from the fact that Δ, being an affine Bruhat-Tits building, is contractible. ∎

Let $\mathcal{A}$ denote the apartment in Δ on which A acts by translations. Let $T = \{g \in G \mid g_{|\mathcal{A}} = \mathrm{id}\}$. T is the maximal compact subgroup of A. $A/T \cong \mathbb{Z}^r$, $r = \mathrm{rank} G = \dim \Delta$.

THE SYMBOLIC SYSTEM. Define a symbolic system Ω:

$$\Omega = \{\omega\colon \mathcal{A} \to Y \mid \omega \text{ is an admissible map}\}.$$

We can view Ω as an r-dimensional subshift of finite type whose symbols space is the set of chambers of Y. The action of A on $\mathcal{A}$ by translations induces an action of A on Ω. Denote the action of $\alpha \in A$ by $S_\alpha\colon \Omega \to \Omega$, $S_\alpha \omega = \omega \circ \alpha_{|\mathcal{A}}$. (We shall refer to a point $\omega \in \Omega$ also as a "tiling".)

Define a map $\varphi\colon \Gamma \backslash G \to \Omega$ by:

$$\varphi(\Gamma g) = \pi \circ g_{|\mathcal{A}}.$$

It is easily seen that φ is well defined, i.e. independent of the choice of $g \in \Gamma g$. As $g\colon \Delta \to \Delta$ and $\pi\colon \Delta \to Y$ are admissible so is $\varphi(\Gamma g)\colon \mathcal{A} \to Y$.

PROPOSITION 2.3: *$\varphi\colon \Gamma\backslash G \to \Omega$ is onto.*

Proof: Let $\omega \in \Omega$. $\omega\colon \mathcal{A} \to Y$ is a proper map. Since $\mathcal{A}$ is simply connected and $\pi\colon \Delta \to Y$ is a covering map, we can lift ω to a map $\tilde{\omega}\colon \mathcal{A} \to \Delta$, s.t. $\pi \circ \tilde{\omega} = \omega$. The admissibility of the maps π and ω implies that $\tilde{\omega}$ is admissible. From the properties of affine buildings (see Proposition 1.1 and Theorem 1.2) it follows that $\tilde{\omega}(\mathcal{A})$ is an apartment. Let $C \in \mathcal{A}$ be a (r-dimensional) chamber. $\tilde{\omega}(C)$ is a chamber in $\tilde{\omega}(\mathcal{A})$. Since G acts strongly transitively on Δ there exists an element $g \in G$ s.t. $g\mathcal{A} = \tilde{\omega}(\mathcal{A})$ and $gC = \tilde{\omega}(C)$. Since both $\tilde{\omega}$ and g preserve the labelling of the building they coincide on C. Hence $g^{-1} \circ \tilde{\omega}$ is an admissible map taking $\mathcal{A}$ to itself fixing a chamber C. Since $\mathcal{A}$ is a thin chamber complex it follows that it is the identity. Hence $g_{|\mathcal{A}} = \tilde{\omega}, \implies \varphi(\Gamma g) = \omega$. ∎

PROPOSITION 2.4: *$\varphi\colon (\Gamma\backslash G, A) \to (\Omega, A)$ is a homomorphism of the dynamical systems.*

Proof: $(\varphi \circ T_\alpha)(\Gamma g) = \varphi(T_\alpha(\Gamma g)) = \varphi(\Gamma g\alpha) = \pi \circ g\alpha_{|\mathcal{A}} = \pi \circ g_{|\mathcal{A}} \circ \alpha_{|\mathcal{A}} = \varphi(\Gamma g) \circ \alpha_{|\mathcal{A}} = S_\alpha(\varphi(\Gamma g)) = (S_\alpha \circ \varphi)(\Gamma g)$. ∎

PROPOSITION 2.5: *φ is continuous.*

Proof: This follows immediately from the continuity of the action of G on Δ. ∎

THE INDUCED MEASURE ON Ω. The Haar measure μ on $\Gamma\backslash G$ induces a measure ν on Ω. Let $\mathcal{F}$ denote the Borel σ-algebra of Ω. For $X \in \mathcal{F}$, let $\nu(X) = \mu(\varphi^{-1}(X))$. Let $R \subset \mathcal{A}$ be a subcomplex, and λ: $R \to Y$ be an admissible map. Define a ("cylindrical") set in Ω:

$$\mathcal{C}_\lambda^R = \left\{\omega \in \Omega \mid \omega_{|_R} = \lambda\right\}.$$

We will omit the superscript R when it is obvious from the context.

PROPOSITION 2.6: *Let $R \subset \mathcal{A}$ be a convex subcomplex, λ_1, λ_2: $R \to Y$ admissible maps. Then*

$$\nu(\mathcal{C}_{\lambda_1}^R) = \nu(\mathcal{C}_{\lambda_2}^R).$$

Proof: First we verify that $\mathcal{C}_{\lambda_i} = \mathcal{C}_{\lambda_i}^R$, $i = 1, 2$, are not empty. Since R is convex it is simply connected, hence there is a lifting $\tilde{\lambda}_i$: $R \to \Delta$ s.t. $\pi \circ \tilde{\lambda}_i = \lambda_i$. An admissible map of a convex subset of $\mathcal{A}$ to Δ is an isometry. Hence $\tilde{\lambda}_i(R) \subset \Delta$ is an isometric image of a subset of an apartment. By Theorem 1.2 it follows that it is contained in an apartment Σ_i of Δ. By the strong transitivity of G, there exist elements $g_i \in G$ such that $g_i\mathcal{A} = \Sigma_i$ and $g_iC = \tilde{\lambda}_i(C)$ for some fixed chamber C in R. Since R is connected, g_i, $\tilde{\lambda}_i$ are admissible maps and apartments are thin, it follows that $g_{i|_R} = \tilde{\lambda}_i$. Hence $\varphi(\Gamma g_i) \in \mathcal{C}_{\lambda_i}$, $i = 1, 2$, $\Rightarrow \mathcal{C}_{\lambda_i} \neq \emptyset$. Let

$$B_i = \{g \in G \mid g_{|_R} = \tilde{\lambda}_i\}.$$

Clearly $\varphi(\Gamma B_i) \subset \mathcal{C}_{\lambda_i}$. Assume that $\omega \in \mathcal{C}_{\lambda_i}$ and $\Gamma g \in \varphi^{-1}(\omega)$. Fix a chamber $C \in R$. For any $g' \in \Gamma g$, $g'C$ is a chamber in Δ satisfying $\pi(g'C) = \pi(\tilde{\lambda}_i(C))$. Since Γ is the covering group of π: $\Delta \to Y$ (Proposition 2.2), there exists $\gamma \in \Gamma$ s.t. $\gamma g'C = \tilde{\lambda}_i(C)$. $\tilde{\lambda}_i$ and $\gamma g'_{|_R}$ are two liftings of λ_i coinciding on C. It follows that $\gamma g'_{|_R} = \tilde{\lambda}_i$; i.e. any element in $\varphi^{-1}(\mathcal{C}_{\lambda_i}) \subset \Gamma\backslash G$ has a representative in B_i. Since Γ acts freely on Δ it follows that this representative is unique. Hence

$$\nu(\mathcal{C}_{\lambda_i}) = \mu(\varphi^{-1}(\mathcal{C}_{\lambda_i})) = \tilde{\mu}(B_i)$$

where $\tilde{\mu}$ is the Haar measure on G normalized so that the measure of a fundamental domain for Γ is 1. Choose $g_1 \in B_1$, $g_2 \in B_2$. It follows that $B_2 = g_2g_1^{-1}B_1$. Since G has a lattice, it is unimodular and $\tilde{\mu}$ is left and right invariant. Hence $\tilde{\mu}(B_1) = \tilde{\mu}(B_2) \Rightarrow \nu(\mathcal{C}_{\lambda_1}) = \nu(\mathcal{C}_{\lambda_2})$. ∎

PROPOSITION 2.7: *For any $\omega \in \Omega$ there exists a $g_0 \in G$ so that $\varphi^{-1}(\omega) = \Gamma g_0 T$. Furthermore, the fiber $\varphi^{-1}(\omega)$ above ω can be identified with T.*

Proof: We have $\varphi^{-1}(\omega) = \{\Gamma g \in \Gamma\backslash G \mid \pi \circ g_{|\mathcal{A}} = \omega\}$. For any $t \in T$, $t_{|\mathcal{A}} = \mathrm{id}$, hence $\varphi^{-1}(\omega)$ is T invariant. Hence $\varphi^{-1}(\omega) = \bigcup_\alpha \Gamma g_\alpha T$, where α belongs to some nonempty indices set. Assume $\Gamma g, \Gamma g' \in \varphi^{-1}(\omega)$. The maps $g_{|\mathcal{A}}, g'_{|\mathcal{A}}\colon \mathcal{A} \to \Delta$ are two liftings of $\omega\colon \mathcal{A} \to Y$. Since Γ is the covering group of $\pi\colon \Delta \to Y$ there exists $\gamma \in \Gamma$ such that $\gamma \circ g'_{|\mathcal{A}} = g_{|\mathcal{A}}$. Hence $g^{-1}\gamma g'_{|\mathcal{A}} = \mathrm{id} \Rightarrow g^{-1}\gamma g' \in T \Rightarrow gT = \gamma g' T \Rightarrow \Gamma g T = \Gamma g' T$. Hence $\varphi^{-1}(\omega) = \Gamma g_0 T$ for some $g_0 \in G$. To verify that the fiber may be identified with T, assume $\Gamma g_0 t = \Gamma g_0 t'$ for some $t, t' \in T$. It follows that $g_0 t' t^{-1} g_0^{-1} \in \Gamma$. Since $t, t' \in T$, it follows that $t' t^{-1}$ and hence also $g_0 t' t^{-1} g_0^{-1}$ has a fixed point in Δ. Since Γ acts freely, $g_0 t' t^{-1} g_0^{-1} = e \Rightarrow t = t'$. ∎

Propositions 2.3, 2.4, 2.5, 2.7 imply:

COROLLARY 2.8: *The map φ induces an isomorphism between the dynamical systems:*

$$\varphi\colon (\Gamma\backslash G/T, A) \to (\Omega, A).$$

COMPACT GROUP/AFFINE EXTENSIONS. Let H be a group acting on a probability space (X, μ). Denote the map corresponding to $h \in H$ by $T_h\colon X \to X$. Let K be a compact group, $\psi\colon H \times X \to K$ a measurable map. Let $h \to \tau_h$ (where $\tau_h\colon K \to K$ is a group automorphism of K) be a representation of H in the group of automorphisms of K. The corresponding affine extension (skew product) is the system $(X \times K, \mu \times \lambda_K, H)$ where λ_K is the Haar probability measure on K. The action of H on $X \times K$ is given by

$$\tilde{T}_h\colon X \times K \to X \times K, \quad \tilde{T}_h(x, k) = (T_h x, \psi(h, x)\tau_h(k)).$$

Notice that in order that $\tilde{T}_h$ will actually define an action of H on $X \times K$, the maps ψ, τ_g, $g \in H$ have to satisfy the following condition: $\psi(gh, x) = \psi(g, T_h x)\tau_g(\psi(h, x))$. When $\tau_h = \mathrm{id}$ for all $h \in H$ the system is called a compact group extension. In this case ψ is a cocycle.

THEOREM 2.1: *$(\Gamma\backslash G, \mu, A)$ is a compact affine extension of $(\Gamma\backslash G/T, \bar{\mu}, A)$. ($\bar{\mu}$ is the measure induced on $\Gamma\backslash G/T$ from the measure μ on $\Gamma\backslash G$.) If A is abelian, this is a compact group extension.*

Proof: Fix a measurable section $f\colon \Gamma\backslash G/T \to \Gamma\backslash G$, i.e., a measurable map F s.t. $\Gamma gT = f(\Gamma gT)T$ for all $\Gamma gT \in \Gamma\backslash G/T$. Next define $\psi\colon A \times \Gamma\backslash G/T \to T$ so that:

$$(*) \qquad f(\Gamma gT)\alpha = f(\Gamma g\alpha T)\psi(\alpha, \Gamma gT)$$

for all $\alpha \in A$, $\Gamma gT \in \Gamma\backslash G/T$. ψ is well defined: $f(\Gamma g\alpha T)T = \Gamma g\alpha T = \Gamma gT\alpha = f(\Gamma gT)T\alpha = f(\Gamma gT)\alpha T$; hence there exists an appropriate $\psi(\alpha, \Gamma gT) \in T$. Moreover it follows from Proposition 2.7 that $\psi(\alpha, \Gamma gT)$ satisfying $(*)$ is unique. Define a map $F\colon \Gamma\backslash G \to \Gamma\backslash G/T \times T$, $F(\Gamma g) = (\Gamma gT, m(\Gamma g))$, where $m\colon \Gamma\backslash G \to T$ satisfies $\Gamma g = f(\Gamma gT)m(\Gamma g)$. It is easy to verify that m is well defined. F is a bijection between $\Gamma\backslash G$ and $\Gamma\backslash G/T \times T$. For $\alpha \in A$ define $\tilde{T}_\alpha\colon \Gamma\backslash G/T \times T \to \Gamma\backslash G/T \times T$ by $\tilde{T}_\alpha(\Gamma gT, t) = (\Gamma g\alpha T, \psi(\alpha, \Gamma gT)\tau_\alpha(t))$, where $\tau_\alpha(t) = \alpha^{-1}t\alpha$ (hence when A is abelian, $\tau_\alpha = \mathrm{id}$ and we will have a compact group extension). We have to verify that $\tilde{T}_\alpha$ defines an action of A on $\Gamma\backslash G/T \times T$. To see this we will show that $F \circ T_\alpha = \tilde{T}_\alpha \circ F$ for all $\alpha \in A$. This will also show that F defines an isomorphism between the dynamical systems $(\Gamma\backslash G, \mu, A)$ and $(\Gamma\backslash G/T \times T, \bar{\mu} \times \lambda_T, A)$. That $F_*(\mu) = \bar{\mu} \times \lambda_T$ follows from the fact that μ is T-invariant and that λ_T is the only T-invariant probability measure on T, and hence on each fiber of the map $\Gamma\backslash G \to \Gamma\backslash G/T$.

$$\begin{aligned}
&F \circ T_\alpha(\Gamma g) = F(\Gamma g\alpha) = (\Gamma g\alpha T, m(\Gamma g\alpha)),\\
&\tilde{T}_\alpha \circ F(\Gamma g) = \tilde{T}_\alpha(\Gamma gT, m(\Gamma g)) = (\Gamma g\alpha T, \psi(\alpha, \Gamma gT)\tau_\alpha(m(\Gamma g))).
\end{aligned}$$

We have:

0. $\psi(\alpha, \Gamma gT), m(\Gamma g), m(\Gamma g\alpha) \in T$.
1. $\Gamma g\alpha = f(\Gamma g\alpha T)m(\Gamma g\alpha)$.
2. $\Gamma g = f(\Gamma gT)m(\Gamma g)$.
3. $f(\Gamma gT)\alpha\psi(\alpha, \Gamma gT)^{-1} = f(\Gamma g\alpha T)$.

2, 3 imply that

$$\begin{aligned}
f(\Gamma g\alpha T)\psi(\alpha, \Gamma gT)\tau_\alpha(m(\Gamma g)) &= f(\Gamma gT)\alpha\psi(\alpha, \Gamma gT)^{-1}\psi(\alpha, \Gamma gT)\tau_\alpha(m(\Gamma g))\\
&= f(\Gamma gT)\alpha\alpha^{-1}m(\Gamma g)\alpha = \Gamma g\alpha.
\end{aligned}$$

From 1 and the uniqueness of $t \in T$ s.t. $f(\Gamma g\alpha T)t = \Gamma g\alpha$, it follows that

$$\psi(\alpha, \Gamma gT)\tau_\alpha(m(\Gamma g)) = m(\Gamma g\alpha).$$

Hence $F \circ T_\alpha(\Gamma g) = \tilde{T}_\alpha \circ F(\Gamma g)$. ∎

The following lemma is a useful tool in studying the subshift of finite type Ω obtained as above:

LEMMA 2.9: *Let $\mathcal{D} \subset \mathcal{A}$ be a connected subcomplex. Let λ: $\mathcal{D} \to Y$ be an admissible map. Denote by $[\mathcal{D}]$ the convex hull of $\mathcal{D}$ in $\mathcal{A}$ (convex hull with respect to the hyperplanes dividing $\mathcal{A}$ into chambers). There is at most one admissible map $[\lambda]$: $[\mathcal{D}] \to Y$ such that $[\lambda]_{|\mathcal{D}} = \lambda$.*

Proof: Let λ_1, λ_2: $[\mathcal{D}] \to Y$ be two admissible maps such that $\lambda_{1|\mathcal{D}} = \lambda = \lambda_{2|\mathcal{D}}$. Lift them to maps $\tilde{\lambda}_i$: $[\mathcal{D}] \to \Delta$ such that $\pi \circ \tilde{\lambda}_i = \lambda_i$ and $\tilde{\lambda}_1(C) = \tilde{\lambda}_2(C)$ for some chamber $C \in \mathcal{D}$. It follows from the connectedness of $\mathcal{D}$ that $\tilde{\lambda}_{1|\mathcal{D}} = \tilde{\lambda}_{2|\mathcal{D}}$. An admissible map of a convex subset of an apartment is an isometry. Hence the image is a convex subset of an apartment. We have $\tilde{\lambda}_1([\mathcal{D}]) = [\tilde{\lambda}_1(\mathcal{D})] = [\tilde{\lambda}_2(\mathcal{D})] = \tilde{\lambda}_2([\mathcal{D}])$. It follows (since apartments are thin and $\tilde{\lambda}_1(C) = \tilde{\lambda}_2(C)$) that $\tilde{\lambda}_1 = \tilde{\lambda}_2$. ∎

COMPACT ORBITS. In this subsection we will show how to use the above construction to obtain information on the distribution of compact A orbits in $\Gamma\backslash G$. For this and the rest of the section we need the assumption that $(\Gamma\backslash G, \mathcal{B}, \mu, A)$ is mixing.

LEMMA 2.10: *Let $\Gamma g \in \Gamma\backslash G$. Its A-orbit, ΓgA, is compact if and only if the A-orbit of $\varphi(\Gamma g) \in \Omega$ is finite (i.e., iff $\varphi(\Gamma g)$ is a periodic tiling).*

Proof: Using the Baire category theorem it follows that an orbit is compact if and only if the stabilizer of a point in the orbit is cocompact. In particular for actions of countable groups, a point in a compact orbit must have a finite index stabilizer and hence the orbit must be finite. Since the A-action on $\Gamma\backslash G/T$ factors via the action of the countable group A/T, it follows that an A-orbit in $\Gamma\backslash G/T$ is compact if and only if it is finite. Thus by Corollary 2.8 an A orbit in Ω is compact if and only if it is finite. Since T is compact it follows that an A-orbit in $\Gamma\backslash G$ is compact if and only if its image in $\Gamma\backslash G/T$, and hence in Ω, is compact, i.e., if and only if the image in Ω is finite. ∎

It follows that if the compact A orbits in $\Gamma\backslash G$ are dense, so are the periodic points in Ω. Observing that an A-orbit in $\Gamma\backslash G$ consists of the fibers (w.r.t. φ) above its image in Ω, and that these fibers, $\varphi^{-1}(\omega)$, are compact, we have that the converse statement also holds:

PROPOSITION 2.11: *If the periodic points in Ω are dense, so are the compact A orbits in $\Gamma\backslash G$.*

Thus in order to show that the compact A orbits in $\Gamma\backslash G$ are dense we have to study the periodic points of Ω. It should be noted that unlike one dimensional subshifts of finite type ("topological Markov shifts") where, under the assumption of transitivity, the periodic points are dense, higher dimensional subshifts of finite type need not, in general, contain periodic points (see for example [Rob]).

The apartment $\mathcal{A}$ is an affine space of dimension r; its partition into chambers is given by a family of hyperplanes. These hyperplanes are translates of finitely many codimension 1 subspaces. Denote the collection of these subspaces by $\mathcal{P}$. The Cartan subgroup A acts on $\mathcal{A}$ as a group of translations via $A/T \cong \mathbb{Z}^r$. Let $O \in \mathcal{A}$ denote the origin. It is easy to see that:

PROPOSITION 2.12: *There exists an element $\alpha \in A$ such that $\alpha O \neq O$ and the line joining O and αO does not lie in any of the subspaces in $\mathcal{P}$.*

Fix a chamber C_0 in $\mathcal{A}$. Let x_0 be a point in its interior. Look at the line L going through x_0 and αx_0. By a small perturbation of the point x_0 we can make sure that L intersects only chambers and their faces of codimension 1. Let $\mathcal{L} \subset \mathcal{A}$ be the subcomplex made of the chambers which L intersects. $\mathcal{L}$ is "a periodic zigzag line". See Figure 2.1 for an example corresponding to $G = \mathrm{PGL}(3, \mathbb{Q}_p)$.

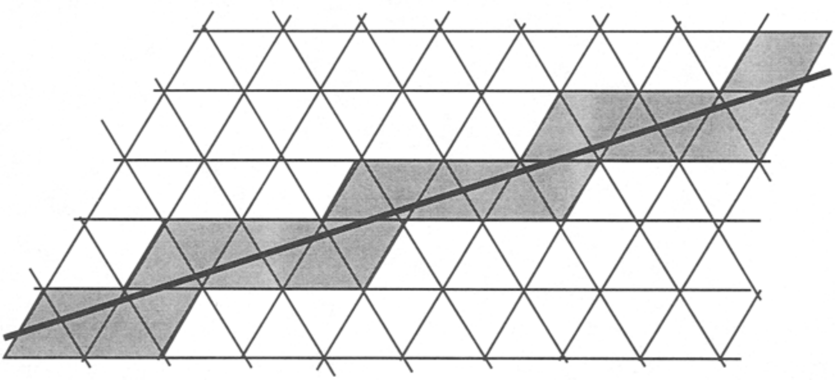

Figure 2.1

Definition: A complex $\mathcal{L}$ as above will be called a **principal gallery**. Notice that the convex hull of a principal gallery is the whole apartment.

Define

$$\Sigma = \Sigma_\alpha = \{f\colon \mathcal{L} \to Y \mid f \text{ admissible}\}.$$

It is easy to see that Σ is a one dimensional subshift of finite type. α acts on $\mathcal{L}$ by translation, inducing a map $S_\alpha: \Sigma \to \Sigma$. Let $R: \Omega \to \Sigma$, $R(\omega) = \omega_{|\mathcal{L}}$.

PROPOSITION 2.13: *The map $R: \Omega \to \Sigma$ is a homeomorphism.*

Proof: i. Injectivity. We have to show that if $\omega, \omega' \in \Omega$ coincide on $\mathcal{L}$, then $\omega = \omega'$. However, as $\mathcal{A}$ is the convex hull of $\mathcal{L}$ this follows immediately from Lemma 2.9.
ii. Surjectivity. Let $f: \mathcal{L} \to Y$ be an admissible map. Lift it to an (admissible) map $\tilde{f}: \mathcal{L} \to \Delta$ (a lifting exists since $\mathcal{L}$ is simply connected). $\tilde{f}$ is an isometry. ($\mathcal{L}$ is a sequence of chambers in an apartment such that moving from chamber to chamber we cross different hyperplanes. Using this and the admissibility of $\tilde{f}$ it can be shown by induction that $\tilde{f}$ is an isometry.) By Theorem 1.2 an isometric image of a part of an apartment is a part of an apartment. Hence there exists an apartment $\mathcal{A}'$ containing $\tilde{f}(\mathcal{L})$. Let $g \in G$ so that $g\mathcal{A} = \mathcal{A}'$ and $gC_0 = \tilde{f}(C_0)$. Let $\omega = f(\Gamma g) \in \Omega$. Clearly $R(\omega) = f$.
iii. Continuity. It is clear that R is continuous. To see that R^{-1} is continuous it is enough to show that for any cylindrical set $\mathcal{C}_\lambda^{\mathcal{D}} \subset \Omega$ where $\mathcal{D} \subset \mathcal{A}$ a finite subcomplex and $\lambda: \mathcal{D} \to Y$ an admissible map, the set $R(\mathcal{C}_\lambda^{\mathcal{D}}) \subset \Sigma$ is open. Since $\mathcal{L}$ is a principal gallery, its convex hull is all of $\mathcal{A}$. Hence there exists a finite "interval" $\bar{\mathcal{L}} \subset \mathcal{L}$ so that its convex hull $[\bar{\mathcal{L}}]$ contains the finite complex $\mathcal{D}$. If $\omega, \omega' \in \Omega$ and $\omega_{|\bar{\mathcal{L}}} = \omega'_{|\bar{\mathcal{L}}}$ then $\omega_{|[\bar{\mathcal{L}}]} = \omega'_{|[\bar{\mathcal{L}}]}$ (see Lemma 2.9). In particular $\omega_{|\mathcal{D}} = \omega'_{|\mathcal{D}}$. It follows that $\mathcal{C}_\lambda^{\mathcal{D}} = \bigcup_i \mathcal{C}_{\lambda_i}^{\bar{\mathcal{L}}}$ where $\{\lambda_i: \bar{\mathcal{L}} \to Y\}$ is the collection of maps $\omega_{|\bar{\mathcal{L}}}: \bar{\mathcal{L}} \to Y$ for $\omega \in \mathcal{C}_\lambda^{\mathcal{D}}$. It is easy to see that $R(\mathcal{C}_{\lambda_i}^{\bar{\mathcal{L}}})$ is an open set in Σ. Hence $R(\mathcal{C}_\lambda^{\mathcal{D}})$ is open. ∎

PROPOSITION 2.14: *Let $\omega \in \Omega$. The A orbit of ω is finite if and only if the orbit of $R(\omega) \in \Sigma$ under S_α is finite.*

Proof: Since the S_α orbit of $R(\omega)$ is contained in the image of the A orbit of ω, if the latter is finite so is the former. Assume that the orbit of $R(\omega)$ is finite, i.e., there exists $d \in \mathbb{N}$ such that $S_\alpha^d R(\omega) = R(\omega)$. It follows from the injectivity of R that

$$S_\alpha^d \omega = \omega. \tag{2.1}$$

Let $\beta_1, \beta_2, \ldots, \beta_r \in A$ so that their images generate $A/T \cong \mathbb{Z}^r$, i.e., they generate the group of translations on Ω. We have to show that for each β_i there exists

$m_i \neq 0$ so that $S_{\beta_i}^{m_i}\omega = \omega$. Fix $\beta = \beta_i$. Look at the sequence of elements $S_\beta^j\omega \in \Omega$. It follows from (2.1) that

$$\begin{aligned} S_\alpha^d(S_\beta^j\omega) &= S_{\alpha^d}S_{\beta^j}\omega = S_{\alpha^d\beta^j}\omega = S_{\beta^j\alpha^d}\omega \\ &= S_{\beta^j}(S_{\alpha^d}\omega) = S_\beta^j(S_\alpha^d\omega) = S_\beta^j\omega, \end{aligned} \tag{2.2}$$

i.e., for any $j \in \mathbb{Z}$, $R(S_\beta^j)$ has an S_α orbit with period $\leq d$. Hence $R(S_\beta^j\omega)$ is determined by the map $S_\beta^j\omega_{|\mathcal{L}_d}: \mathcal{L}_d \to Y$, where $\mathcal{L}_d$ is an "interval" of d chambers in $\mathcal{L}$. Since R is injective it follows that $S_\beta^j\omega$ is determined $S_\beta^j\omega_{|\mathcal{L}_d}$. Since the collection of these maps is finite it follows that there exist $k \neq l \in \mathbb{Z}$ such that $S_\beta^k\omega = S_\beta^l\omega$. Hence $S_\beta^m\omega = \omega$ for $m = k - l \neq 0$. The A orbit of ω is finite. ∎

PROPOSITION 2.15: *The system (Σ, S_α) is topologically transitive.*

Proof: The system (Σ, S_α) is a factor of the system $(\Gamma\backslash G, T_\alpha)$ (the factor map being $R \circ \varphi: \Gamma\backslash G \to \Sigma$). The measure μ on $\Gamma\backslash G$ induces a measure $\bar\nu$ on Σ. Identifying Ω and Σ via the homeomorphism R, the measure ν is identified with $\bar\nu$. It follows from Proposition 2.6 that for any nonempty open set $U \subset \Sigma$ we have $\bar\nu(U) > 0$. Using the assumption that $(\Gamma\backslash G, \mathcal{B}, \mu, A)$ is mixing and in particular $T_\alpha: \Gamma\backslash G \to \Gamma\backslash G$ is mixing, we conclude that $(\Sigma, \bar\nu, S_\alpha)$ is mixing. Hence if $U, V \subset \Sigma$ are nonempty open sets then $\bar\nu(U), \bar\nu(V) > 0$ and hence $\lim_{n\to\infty} \bar\nu({S_\alpha}^n U \cap V) = \bar\nu(U)\bar\nu(V) > 0$. In particular ${S_\alpha}^n U \cap V \neq \emptyset$ for large enough n. ∎

PROPOSITION 2.16: *The measure $\bar\nu$ on Σ is the unique invariant measure so that $h_{\bar\nu}(S_\alpha) = h_{\mathrm{top}}(S_\alpha)$.*

We postpone the proof of this proposition to the next subsection (see Corollary 2.20).

THEOREM 2.2':

(1) *The periodic points in Ω are dense.*
(2) *Let $\alpha \in A$ be as in Proposition 2.12. Let $\tilde{\mathcal{P}}_N \subset \Omega$ be the set of points whose orbit under S_α has size $\leq N$. $\tilde{\mathcal{P}}_N$ is a finite set. Let $\tilde\zeta_N$ be the normalized counting measure on $\tilde{\mathcal{P}}_N$. Then $\lim_{n\to\infty} \tilde\zeta_N = \nu$ (in the weak* topology).*
(3) *Let $a_n = \#\{\omega \in \Omega \mid {S_\alpha}^n\omega = \omega\}$. Define*

$$F_\alpha(x) = \exp\left(\sum_1^\infty \frac{a_n}{n}x^n\right).$$

Then

$$F_\alpha(x) = \frac{1}{\det(I - xM_\alpha)},$$

where M_α is the adjacency matrix of the subshift of finite type Σ_α defined above.

Proof: Using Propositions 2.13, 2.14 and 2.15, (1) follows from the well known result that the periodic points of a topologically transitive one-dimensional subshift of finite type are dense. (See the proof of Proposition 17.13, page 128 in [DGS].) In light of Proposition 2.16, (2) is just a special case of the assertion that in an aperiodic one dimensional subshift of finite type the sequence of normalized counting measures supported on points of period $\leq N$ converges to the measure of maximal entropy as N tends to infinity. (3) is well known in connection with the rationality of the zeta function associated with periodic points of a subshift of finite type. ∎

THEOREM 2.2: *Let G, Γ, A, μ be as above, and assume that $(\Gamma\backslash G, \mathcal{B}, \mu, A)$ is mixing. Then:*

1. *The compact A orbits in $\Gamma\backslash G$ are dense.*
2. *Let ζ be the Haar measure on A normalized so that $\zeta(T) = 1$ (T is the maximal compact subgroup of A). Let $\alpha \in A$ be as in Proposition 2.12. ζ induces a measure on compact orbits of $\langle\alpha, T\rangle$ in $\Gamma\backslash G$. Let $\mathcal{P}_N \subset \Gamma\backslash G$ be the set of points whose orbit under $\langle\alpha, T\rangle$ is compact and has measure $\leq N$. $\mathcal{P}_N$ is made of finitely many $\langle\alpha, T\rangle$ orbits. Normalize the measure induced on it from ζ to get a probability measure denoted by ζ_N. Then $\lim_{n\to\infty} \zeta_N = \mu$ (in the weak* topology).*
3. *Let $a_n = \zeta(\{\Gamma g \in \Gamma\backslash G \mid \Gamma g\alpha^N \in \Gamma g T\})(= \#\{\omega \in \Omega \mid {S_\alpha}^n\omega = \omega\})$. Define*
$$F_\alpha(x) = \exp\left(\sum_1^\infty \frac{a_n}{n}x^n\right).$$
Then
$$F_\alpha(x) = \frac{1}{\det(I - xM_\alpha)},$$
where M_α is the adjacency matrix of the subshift of finite type Σ_α defined above.

Proof: The assertions follows from the corresponding ones in Theorem 2.2'. (1) follows using Proposition 2.11, (2) follows using Theorem 2.1. ∎

BERNOULLICITY. $g_0 \in G$ is said to have an **axis** if there exist an apartment $\mathcal{A}_0$ and a line $L \subset \mathcal{A}_0$ so that L is g_0 invariant and g_0 acts on L by translating it a distance $d_0 > 0$. Notice that any $\alpha \in A$ which generates a noncompact subgroup has an axis. We will assume throughout this subsection that the system $(\Gamma\backslash G, \mathcal{B}, \mu, T_{g_0})$ is mixing. We will show that under these assumptions this system is Bernoulli. Fix a chamber $C_0 \in \mathcal{A}_0$ such that $L \cap C_0 \neq \emptyset$.

PROPOSITION 2.17: *There exists $r \in \mathbb{N}$ so that $g_0^r C_0 \in \mathcal{A}_0$ and is a translate of C_0: $g_0^r C_0 = C_0 + v_0$, where v_0 is a nonzero vector in the linear space corresponding to $\mathcal{A}_0$ parallel to the line L.*

Proof: Look at the collection of apartments containing the line L. Restrict our attention to the intersections of each such apartment with the collection of chambers touching L. We obtain a finite collection (recall that the building Δ is locally finite) of "strips". Since g_0 preserves L as well as the collection of apartments containing L, it follows that it induces a permutation on this finite collection of strips. Hence some power of it maps each such strip, in particular the one coming from $\mathcal{A}_0$, to itself. Taking some further power (bounded by a function of the dimension of the building) of it we conclude that its action on C_0 is by translation within $\mathcal{A}_0$. ∎

By a theorem of D. Ornstein (see [O1], Theorem 4, page 39), in order to prove that T_{g_0} is Bernoulli it is enough to show that $T_{g_0^r} = {T_{g_0}}^r$ is Bernoulli. Hence we will assume that already g_0 acts on a chamber $C_0 \in \mathcal{A}_0$ as translation by a vector v_0. We will also be interested in computing entropies of elements having an axis. Since $\mathrm{h}(T_{g_0}^r) = r\mathrm{h}(T_{g_0})$ it will be enough to consider such g_0. We can assume now without loss of generality that the line L passes through the interior of C_0 and intersects only the interior and codimension one faces of chambers. Let $\mathcal{L} \subset \mathcal{A}_0$ be the subcomplex consisting of the chambers intersected by L. Define a subshift of finite type: $\Sigma = \{f \colon \mathcal{L} \to Y \mid f \text{ admissible}\}$, $g_0 \colon \mathcal{L} \to \mathcal{L}$ induces a map $S_{g_0} \colon \Sigma \to \Sigma$. Let $\psi \colon \Gamma\backslash G \to \Sigma$ be given by $\psi(\Gamma g) = \pi \circ g_{|\mathcal{L}}$. $\mathcal{F}$ denotes the Borel σ-algebra of Σ, η the induced measure on Σ from μ via ψ. The following proposition is proved in the same way as Propositions 2.4, 2.8 and Theorem 2.1 (and coincide with them when g_0 is generic).

PROPOSITION 2.18:

(1) $\psi \colon (\Gamma\backslash G, \mathcal{B}, \mu, T_{g_0}) \to (\Sigma, \mathcal{F}, \eta, S_{g_0})$ *is an epimorphism.*

(2) *Let $M < G$ be the subgroup $M = \{g \in G \mid g_{|\mathcal{L}} = \mathrm{id}\}$. Then M is compact and the fibers $\psi^{-1}(f)$ are M orbits.*

(3) *The system $(\Gamma\backslash G, \mathcal{B}, \mu, T_{g_0})$ is a compact affine extension of $(\Sigma, \mathcal{F}, \eta, S_{g_0})$.*

Denote by $\mathcal{C}_\lambda^R \subset \Sigma$ the cylindrical subset corresponding to an admissible map $\lambda\colon R \to Y$ where $R \subset \mathcal{L}$ a subcomplex. Then in the same way as in Proposition 2.6 we have:

PROPOSITION 2.19: *Let $R \subset \mathcal{L}$ and $\lambda_1, \lambda_2\colon R \to Y$ be admissible maps, then $\eta(\mathcal{C}_{\lambda_1}^R) = \eta(\mathcal{C}_{\lambda_2}^R)$.*

$\mathcal{L}$ is a sequence of chambers on which g_0 acts by translation. Let $\mathcal{D} \subset \mathcal{L}$ be a connected subcomplex which is a fundamental domain for this action. $\mathcal{D}$ is a finite subcomplex. $\mathcal{L} = \biguplus_{i\in\mathbb{Z}} g_0^i\mathcal{D}$. A map $\Sigma \ni f\colon \mathcal{L} \to Y$ induces a sequence of maps $(f_i)_{i\in\mathbb{Z}}$, $f_i = f_{|\mathcal{D}_i}\colon \mathcal{D}_i \to Y$, where $\mathcal{D}_i = g_0^i\mathcal{D}$. Denote

$$\tilde{\Sigma} = \{(f_i)_{i\in\mathbb{Z}} \mid f_i\colon \mathcal{D}_i \to Y \text{ induces an admissible map } f\colon \mathcal{D} \to Y\}.$$

$\tilde{\Sigma}$ is a one dimensional subshift of finite type. (Its alphabet is the set of admissible maps $\bar{f}\colon \mathcal{D} \to Y$.) It is naturally identified with Σ. Let $\tilde{S}\colon \tilde{\Sigma} \to \tilde{\Sigma}$ be the shift transformation. It corresponds to S_{g_0}. Let $\tilde{\nu}$ be the measure on $\tilde{\Sigma}$ corresponding to ν via this identification. From Proposition 2.19 it follows that cylindrical sets corresponding to legal words of equal length have the same measure.

COROLLARY 2.20: *$\tilde{\nu}$ is a maximal entropy Markovian measure on $\tilde{\Sigma}$.*

Proof: $\tilde{\nu}$ is stationary. To check that it is Markov, consider the cylindrical sets $\mathcal{C}_{f_i\dots f_j}$, $i < j$, and $\mathcal{C}_{f_i\dots f_j f_{j+1}^s}$ where $\{f_i \dots f_j f_{j+1}^s \mid 1 \le s \le t\}$ is the collection of all legal words of length $j-i+2$ starting with $f_i \dots f_j$. Their number, t, depends only on f_j. They all have the same measure: $\tilde{\nu}(\mathcal{C}_{f_i\dots f_j f_{j+1}^s}) = \tilde{\nu}(\mathcal{C}_{f_i\dots f_j f_{j+1}^{s'}})$. Hence

$$\frac{\tilde{\nu}(\mathcal{C}_{f_i\dots f_j f_{j+1}^s})}{\tilde{\nu}(\mathcal{C}_{f_i\dots f_j})} = \frac{1}{t}.$$

Note that if the alphabet set $\{\bar{f}\colon \mathcal{D} \to Y\}$ is of size d then $\tilde{\nu}(\mathcal{C}_{\bar{f}}) = 1/d$, for any admissible $\bar{f}\colon \mathcal{D} \to Y$. Hence $\tilde{\nu}$ is a Markov measure. To see that it is of maximal entropy, notice that since cylindrical sets corresponding to legal words of given length have the same measure, it follows that the number (t) of symbols allowed to follow (or precede) any given symbol is independent of the symbol. Hence the Markov measure $\tilde{\nu}$, which corresponds to the adjacency matrix divided by t, is the (unique) measure of maximal entropy. ∎

PROPOSITION 2.21: *The system* $(\Sigma, \mathcal{F}, \nu, S_{g_0})$ *is Bernoulli.*

Proof: $(\Sigma, \mathcal{F}, \nu, S_{g_0})$ is identified with $(\tilde{\Sigma}, \tilde{\mathcal{F}}, \tilde{\nu}, \tilde{S})$. The latter is Markov by Proposition 2.20. We have assumed that the system $(\Gamma\backslash G, \mathcal{B}, \mu, T_{g_0})$ is mixing. Hence also its factor $(\Sigma, \mathcal{F}, \nu, S_{g_0})$ (and $(\tilde{\Sigma}, \tilde{\mathcal{F}}, \tilde{\nu}, \tilde{S})$) is mixing. By a theorem of N. Friedman and D. Ornstein ([F-O]) a mixing Markov system is Bernoulli. ∎

THEOREM 2.3: *The system* $(\Gamma\backslash G, \mathcal{B}, \mu, T_{g_0})$ *is Bernoulli.*

Proof: It is a mixing compact affine extension of a Bernoulli system (see Propositions 2.18 and 2.21). (In case it is actually a compact group extension, then by a theorem of D. Rudolph ([Ru]) it is Bernoulli.) The compact group M, appearing in the affine extension, is totally disconnected. By the results of Juzvinskii and of Thomas (see [Th1], [Ro1], [Ro2], [W, section 2]) the affine extension is the limit of a sequence of affine extensions each of which is either by a finite group or such that the group automorphism is Bernoulli. Since all these intermediate extensions are mixing (being factors of the original mixing system), it follows by using Rudolph's result or the results of D. Lind (see [Li]) that they, and hence $(\Gamma\backslash G, \mathcal{B}, \mu, T_{g_0})$, are Bernoulli systems. ∎

Next we examine the (directional) entropies $\mathrm{h}_\mu(T_{g_0})$. By Thomas' "Addition theorem" (see [Th2]) $\mathrm{h}_\mu(T_{g_0}) = \mathrm{h}_{\tilde{\nu}}(\tilde{S}) + \mathrm{h}(\tau)$, where τ is the automorphism of the group M. Look at the sequence of chambers in $\mathcal{L}$ connecting the chamber C_0 to g_0C_0. Denote it by $C_0, C_1, \ldots, C_d = g_0C_0$. Let $k_i + 1$ be the number of chambers containing the face $C_i \cap C_{i+1}$.

THEOREM 2.4:

$$h_{\tilde{\nu}}(\tilde{S}) = \sum_{i=0}^{d-1} \log k_i.$$

Proof: We have to show $\mathrm{h}_{\tilde{\nu}}(\tilde{S}) = \sum_{i=0}^{d-1} \log k_i$. $(\tilde{\Sigma}, \tilde{\mathcal{F}}, \tilde{\nu}, \tilde{S})$ is a Markov system in which every symbol has the same number, t, of successive symbols, and the transition probabilities are $1/t$. Hence $\mathrm{h}_{\tilde{\nu}}(\tilde{S}) = \log t$. It is easy to see that t is the number of admissible maps of the complex $C_0, C_1, \ldots, C_d$ into Δ fixing C_0. The number of possible maps of C_1 is k_0. Similarly, after determining the images of $C_0, C_1, \ldots, C_i$ there are k_i possible ways of mapping C_{i+1}. Hence $t = k_0K_1 \ldots k_{d-1}$. (Note that the image of the face $C_i \cap C_{i+1}$ belongs to $K_i + 1$

chambers because there is an automorphism of the building whose restriction to $C_0, C_1, \ldots, C_i$ coincides with the admissible map we have chosen so far.) ∎

We remark that the above formula holds also for elements which fix C_0. In this case the entropy is 0. We assume now that A is abelian and examine the structure of the entropy function: $\mathrm{h}_\mu: A \to \mathbb{R}^+$ in the system $(\Gamma\backslash G, \mathcal{B}, \mu, A)$. The apartment $\mathcal{A}$ is the Coxeter complex corresponding to an affine Weyl group W. Let $\overline{W}$ be the corresponding spherical Weyl group. $\overline{W}$ is a finite group. Its Coxeter complex is a decomposition of $\mathcal{A}$ into finitely many convex cones, based at a common vertex, denoted O — the origin. This decomposition induces a decomposition of A into "Weyl chambers". Two elements $a_1, a_2 \in A$ are said to belong to the same Weyl chamber if the corresponding images of O: a_1O, a_2O belong to the same cone in the Coxeter complex of $\overline{W}$. Note that this decomposition is not disjoint and the chambers intersect at their boundaries.

THEOREM 2.5: *The function $h_\mu: A \to \mathbb{R}^+$ is piecewise linear: its restriction to a Weyl chamber in A is linear.*

Proof: Since we assume that A is abelian, the system $(\Gamma\backslash G, \mathcal{B}, \mu, A)$ is a compact group extension of the subshift of finite type $(\Omega, \mathcal{F}, \eta, A)$ and the entropy is the same as the entropy of the action on Ω. Let $\mathcal{L}$ be a subcomplex defined as above for an element of A. $\mathcal{L}$ is contained in $\mathcal{A}$ and Theorem 2.4 gives a formula for the entropy of the factor system obtained by restricting to $\mathcal{L}$. This is, however, equal to the entropy of the action on Ω, as is easily seen using wider and wider neighbourhoods of $\mathcal{L}$ in $\mathcal{A}$ and Lemma 2.9. The numbers k_i appearing in Theorem 2.4 may be defined also as follows. Let $P_0, P_1, \ldots, P_{d-1}$ be the hyperplanes separating the chambers C_0 and g_0C_0 in $\mathcal{A}$. $k_i + 1$ is the number of chambers in Δ which contain a codimension 1 face contained in P_i. K_i is well defined independent of the particular face. Fix some chamber in $\mathcal{A}$ containing the given face and the half apartment of $\mathcal{A}$ containing it bounded by P_i. For any other chamber containing this face choose an apartment containing it as well as the above half apartment. We obtain, say, k apartments, one for each adjacent chamber. Note that the intersection of any two of these apartments is exactly the above half apartment. It follows that any chamber in this half apartment which touches P_i has at least k neighbours via its face in P_i. Hence the number $k = k_i$ is independent of the specific face. Consider now two elements $a_1, a_2 \in A$ belonging to the same Weyl chamber. They act on $\mathcal{A}$ as translations by vectors v_1, v_2 respectively. The assumption that they belong to the same Weyl chamber in A means that for any of the linear functionals $\varphi_i\colon \mathcal{A} \to \mathbb{R}$ defining the hyperplanes partitioning $\mathcal{A}$,

$$\varphi_i(a_1 O)\varphi_i(a_2 O) \geq 0. \tag{2.3}$$

It follows that if $P_0, P_2, \ldots, P_{d-1}$ are the hyperplanes separating C_0 from $a_1 C_0$ and $Q_0, Q_1, \ldots, Q_{l-1}$ are the hyperplanes separating C_0 from $a_2 C_0$, then:

(1) $Q_0 + v_1, \ldots, Q_{l-1} + v_1$are the hyperplanes separating $a_1 C_0$ from $a_2 a_1 C_0$.

(2) $P_0, \ldots, P_{d-1}, Q_0 + v_1, \ldots, Q_{l-1} + v_1$ are the hyperplanes separating C_0 from $a_2 a_1 C_0$.

To complete the proof of the assertion that $\mathrm{h}_\mu(T_{a_1 a_2}) = \mathrm{h}_\mu(T_{a_1}) + \mathrm{h}_\mu(T_{a_2})$, note that since a_1 maps a face in Q_i to a face in $Q_i + v_1$ and a_1 is an automorphism of the building, the number of chambers containing a given face in Q_i equals the number of chambers containing a given face in $Q_i + v_1$. ∎

3. Examples and applications

p-ADIC CHEVALLEY GROUPS. N. Iwahori and H. Matsumoto, [I-M], have shown that a semisimple Chevalley group over $\mathbb{Q}_p$, G, has a BN pair so that the associated building is an affine building. G acts strongly transitively on the complete apartment system of the building. Hence the results of the previous section apply to this case. We note that the Howe-Moore theorem on vanishing of matrix coefficients implies that for G a semisimple Chevalley group over $\mathbb{Q}_p$, the action of the split Cartan subgroup on $\Gamma\backslash G$ for an irreducible lattice $\Gamma < G$ is mixing (see [Z]). The Cartan subgroup A in these cases is abelian. We shall also use the following well known results:

THEOREM (see [Tam]): *A lattice in a semisimple Chevalley group over $\mathbb{Q}_p$ is uniform (i.e., cocompact).*

THEOREM (see [Sel]): *If G a semisimple Chevalley group over $\mathbb{Q}_p$ and $\Gamma < G$ a uniform lattice, then there exists a finite index sublattice $\Gamma_0 < \Gamma$ having no torsion.*

Thus we have:

THEOREM 3.1: *Let G be a semisimple Chevalley group over $\mathbb{Q}_p$, $\Gamma < G$ an irreducible torsion free lattice, $A < G$ a split Cartan subgroup of G, $\mathcal{B}$ the Borel σ-algebra of $\Gamma\backslash G$, μ Haar probability measure on $\Gamma\backslash G$, $T < A$ the maximal compact subgroup of A.*

(1) *Let $H < A$ be a closed subgroup, $d = \mathrm{rank} H$ (i.e., $H/H \cap T \cong \mathbb{Z}^d$). Then there exists a d-dimensional subshift of finite type $(\Omega, \mathcal{F}, \nu, H)$ on which H acts via $H/H \cap T$ so that $(\Gamma\backslash G, \mathcal{B}, \mu, H)$ is a compact affine extension of $(\Omega, \mathcal{F}, \nu, H)$. When H contains a regular element, the extension is a compact group extension.*

(2) *The compact A orbits in $\Gamma\backslash G$ are dense.*

(3) *Let ${\mathbb{Q}_p}^* \cong H' < A$ be a regular one parameter subgroup, let $H = TH'$, η Haar measure on H normalized so that $\eta(T) = 1$. η induces a measure on compact H orbits in $\Gamma\backslash G$. The measure of a compact H orbit is a natural number. Denote by a_n the number of compact orbits of measure n. $a_n < \infty$, and*

$$\exp \sum_{n=1}^{\infty} \sum_{d|n} \frac{d a_d}{n} x^n = \frac{1}{\det(I - xM)}$$

where M is the adjacency matrix of a corresponding one-dimensional subshift of finite type.

(4) *(Notations as in (3).) Let μ_N be the probability measure obtained by normalizing the sum of the measures induced from η on the compact H orbits of measure $\leq N$. Then $\lim_{N\to\infty} \mu_N = \mu$ in the weak* topology.*

(5) *For any $g \in A$ such that $< g >$ is not compact, the (one-dimensional) system $(\Gamma\backslash G, \mathcal{B}, \mu, T_g)$ is Bernoulli.*

(6) *The directional entropy function h_μ: $A \to \mathbb{R}^+$ is piecewise linear.*

Proof: The assertions follow from Theorems 2.1, 2.2, 2.3, 2.4, 2.5 applied to a group G as above. Actually we have shown (1) in the previous section only for subgroups H which are either the whole of A or one parameter subgroups of A; the treatment of the more general case is a simple adaptation of those cases and is omitted. ∎

By Selberg's theorem any lattice $\Gamma < G$ has a torsion free, finite index sublattice $\Gamma_0 < \Gamma$. Hence the above theorem applies to the system $(\Gamma_0\backslash G, \mathcal{B}_0, \mu_0, A)$. There is a natural finite to one covering map $\Gamma_0\backslash G \to \Gamma\backslash G$. It follows that

(2') The compact A orbits in $(\Gamma\backslash G, \mathcal{B}, \mu, A)$ are dense.

(5') For any $g \in A \setminus T$, $(\Gamma\backslash G, \mathcal{B}, \mu, T_g)$ is Bernoulli.

(6') The directional entropy function $\mathrm{h}_\mu\colon A \to \mathbb{R}^+$ is piecewise linear.

In the case of a semisimple Chevalley group over $\mathbb{Q}_p$ the formula for the directional entropy (Theorem 2.4) is especially simple. We shall need the following result of N. Iwahori and H. Matsumoto.

THEOREM (see [I-M] proposition 2.6): *Let G be a semisimple Chevalley group over $\mathbb{Q}_p$, Δ its affine building. Then every codimension 1 face of Δ belongs to exactly $p+1$ chambers.*

Combined with Theorem 2.4 we have for $g \in A$ that $\mathrm{h}_\mu(T_g) = \lambda(g)\log p$, where $\lambda(g)$ is the number of hyperplanes separating a chamber $C \in \mathcal{A}$ from its translate gC.

CLAIM (see [I-M] proposition 1.10): *For all $g \in A$, $\lambda(g)$ equals the length $l(g)$ of the image of g in the affine Weyl group $N(A)/T$ with respect to the standard generators.*

COROLLARY 3.1: *For $g \in A$, $\mathrm{h}_\mu(T_g) = l(g)\log p$.*

N. Iwahori and H. Matsumoto give a formula for $l(g)$ (see [I-M] section I.9, p. 20). We will use a slightly modified form of this formula since we prefer viewing the roots of (G, A) as homomorphisms from A to ${\mathbb{Q}_p}^*$. Notice that the formula is based on the fact that one can interpret the roots as linear functionals on the apartment $\mathcal{A}$ and the hyperplanes separating $\mathcal{A}$ into chambers are the hyperplanes where these functionals have integral values.

PROPOSITION 3.2 (see [I-M] I.9): *Let Φ be the root system of (G, A), $\Phi_+ \subset \Phi$ the positive roots. Then*

$$l(g)\log p = \lambda(g)\log p = \sum_{\varphi\in\Phi_+} |\log|\varphi(g)|_p| = \sum_{\varphi\in\Phi} \log^+ |\varphi(g)|_p$$

where $|x|_p = p^{-\nu(x)}$ is the p-adic absolute value of x, and $\log^+ y = \log y$ for $y > 1$ and 0 otherwise.

COROLLARY 3.3: *For $g \in A$, $\mathrm{h}_\mu(T_g) = \sum_{\varphi\in\Phi} \log^+ |\varphi(g)|_p$.*

Let H be a semisimple Lie group over $\mathbb{R}$ and Φ_H its root system with respect to some Cartan subgroup $A_H < H$. Notice that for $h \in A_H$, $\{\varphi(h) \mid \varphi \in \Phi_H\}$ are the eigenvalues of $\mathrm{Ad}(h)$ acting on the Lie algebra of H. It follows that these are the Lyapunov exponents of the system $(\Gamma_H\backslash H, \mathcal{B}_H, \mu_H, T_h)$ (where $\Gamma_H < H$ is a lattice, and $T_h\colon \Gamma_H\backslash H \to \Gamma_H\backslash H$ translation by h). Hence by Pesin's formula (see [Mañé], Corollary 10.3, p. 265):

$$\mathrm{h}_{\mu_H}(T_h) = \sum_{\varphi\in\Phi_H} \log^+ |\varphi(h)|.$$

So the formula in Corollary 3.3 is a p-adic analog of the formula obtained in the real case via Pesin's theorem.

We examine now some specific groups. Let $G = \mathrm{PSL}(2, \mathbb{Q}_p)$. It is a rank one group and the corresponding affine building Δ is a $p+1$-regular tree (see [Ser]). An apartment is an infinite line in the tree. For a torsion free lattice $\Gamma < G$ the quotient complex $Y = \Gamma\backslash\Delta$ is a finite $p+1$-regular graph. Both Δ and Y are bipartite (2-colorable). Fix some 2-coloring. The subshift of finite type we obtain for this case is the space of all bi-infinite paths in the graph Y with no "folding" and such that at time 0 the path visits a vertex of a fixed color. The case of $\mathrm{PSL}_2(\mathbb{R})$ corresponds to the familiar geodesic flow on a Riemann surface. The fact that the geodesic flow is Bernoulli was proved by D. Ornstein and B. Weiss (see [O-W], [O1]).

Let $G = \mathrm{PSL}_3(\mathbb{Q}_p)$. It is a rank 2 group. The corresponding building is a two dimensional simplicial complex. An apartment is isomorphic to a plane tesselated by equilateral triangles.

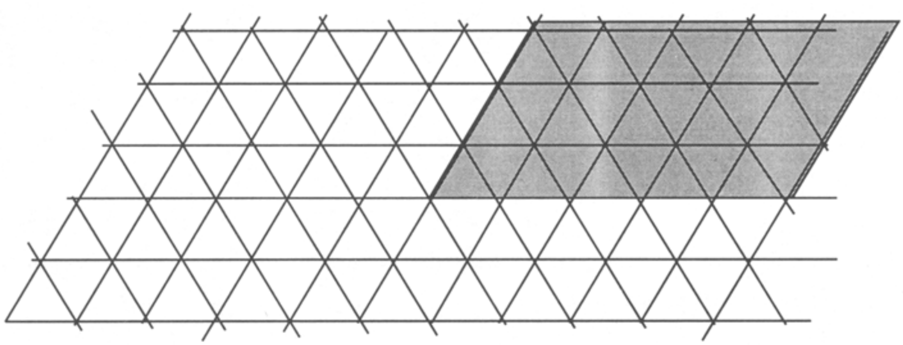

Figure 3.1

Each edge belongs to $p+1$ triangles. The Cartan subgroup is conjugate to

$$A = \left\{ \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix} \in G \right\}.$$

Denote by $T = \left\{ \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix} \in A \mid \lambda_1, \lambda_2, \lambda_3 \in \mathbb{Z}_p{}^* \right\}$. The spherical Weyl group of G is S_3. A is decomposed into 6 Weyl chambers: $\tau \in S_3$;

$$A_\tau = \left\{ \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix} \in A \mid \nu_p(\lambda_{\tau(1)}) \leq \nu_p(\lambda_{\tau(2)}) \leq \nu_p(\lambda_{\tau(3)}) \right\}$$

($\nu_p(\cdot)$ is the p-adic valuation). See Figure 3.1.

We describe two applications of the tools developed above relating to $G = \mathrm{PSL}(3, \mathbb{Q}_p)$.

Closures of A orbits. We have seen that there are many compact A-orbits in $\Gamma\backslash G$. An interesting question raised by H. Furstenberg and by G. A. Margulis is what are the possible closed A-invariant subsets of $\Gamma\backslash G$. There has been a lot of work concerning the possible closures of orbits of subgroups generated by unipotents in homogeneous spaces of a real Lie group. M. Ratner has proved the Raghunathan conjecture which states that such a closure is again an orbit, but of a possibly larger subgroup. (See [Ra], [Marg1], [Marg2], [D-M].) It is well known that for $G = \mathrm{PSL}(2, \mathbb{R})$ and $A < G$ its Cartan subgroup, there are orbits whose closure is not even a manifold. G. A. Margulis has conjectured in his ICM90 talk ([Marg2]) that for higher rank (real Lie) groups, if the closure, in $\Gamma\backslash G$, of an orbit under the Cartan is compact, then it is already an orbit, under the additional condition that the lattice Γ does not contain any semisimple element with multiple eigenvalue. The necessity of such a condition was shown by M. Rees ([Re]). We will indicate here how one can see the necessity of such a condition also for $G = \mathrm{PSL}(3, \mathbb{Q}_p)$ using the symbolic description developed above.

Let $\Gamma < G = \mathrm{PSL}(3, \mathbb{Q}_p)$ be a lattice. Assume that there exists a semisimple element $\gamma_0 \in \Gamma$ which has a multiple eigenvalue. By examining the characteristic polynomial of γ_0 it is not hard to see that either γ_0 or γ_0^2 is diagonalizable over $\mathbb{Q}_p$. We assume, w.l.o.g., that $\gamma_0 = g\alpha_0 g^{-1}$ where

$$\alpha_0 = \begin{pmatrix} \lambda & & \\ & \lambda & \\ & & \lambda^{-2} \end{pmatrix} \in A.$$

Let Ω be the two dimensional subshift of finite type associated with $(\Gamma\backslash G, \mathcal{B}, \mu, A)$. α_0 induces on $\mathcal{A}$ as a shift parallel to one of its tesselation lines.

Let $\Omega_0 = \{\omega \in \Omega \mid S_{\alpha_0}\omega = \omega\}$. It is readily seen that Ω_0 is a subshift of finite type. It follows from the commutativity of A that Ω_0 is invariant under A. A acts on it via $A/T \cong \mathbb{Z}^2$. Since α_0 acts trivially on Ω_0, it follows that A acts on Ω_0 via $A/ < T, \alpha_0 > \cong \mathbb{Z} \times \mathrm{C}_r$ where C_r is a cyclic group of order r. Let $\bar{\Omega}_0 = \Omega_0/\mathrm{C}_r$ be the quotient system.

LEMMA 3.4: *$\bar{\Omega}_0$ is a one dimensional subshift of finite type containing infinitely many points.*

Proof: The fact that $\bar{\Omega}_0$ is a one dimensional subshift of finite type follows from the preceding discussion. For $t \in \mathbb{Q}_p$ let

$$u_t = \begin{pmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Since $\gamma_0 = g\alpha_0 g^{-1}$, we have $\Gamma g\alpha_0 = \Gamma g$. Hence the image of Γg in Ω is in Ω_0, hence giving an element in $\bar{\Omega}_0$. Moreover, since $u_t\alpha_0 = \alpha_0 u_t$ it follows that $\Gamma g u_t \alpha_0 = \Gamma g u_t$ for every $t \in \mathbb{Q}_p$ and each of these defines an element of $\bar{\Omega}_0$. Examining the way u_t acts on $\mathcal{A}$ (which is by fixing half a plane bounded by a line parallel to the translation direction of α_0's action) one sees that all these elements are distinct. ∎

Notice that if a one dimensional subshift of finite type contains infinitely many points, then it has points whose orbit closure has arbitrarily small Hausdorff dimension. Now take such a point $\bar{\omega}$ and look at a preimage Γx of it in $\Gamma\backslash G$. The closure of $\Gamma x A$ is mapped to the closure of the A-orbit of $\bar{\omega}$. Since there are only finitely many possible Hausdorff dimensions for closed orbits of subgroups of G, it follows that we can choose $\bar{\omega}$ so that the obtained orbit closure is not an orbit.

The second application we give is the following:

PROPOSITION 3.5: *There exist two two-dimensional subshifts of finite type, $(\Omega, \mathcal{F}, \nu, \mathbb{Z}^2)$, $(\Sigma, \mathcal{M}, \eta, \mathbb{Z}^2)$, so that for every $v \in \mathbb{Z}^2$ the corresponding one-dimensional systems are both isomorphic Bernoulli systems, but the systems are not isomorphic as $\mathbb{Z}^2$ systems.*

Proof: The first system, $(\Omega, \mathcal{F}, \mu, \mathbb{Z}^2)$, is the two-dimensional subshift constructed for $G = \mathrm{PSL}(3, \mathbb{Q}_2)$, $\Gamma < G$ any uniform lattice and $A < G$ the Cartan subgroup. $\mathbb{Z}^2$ acts on Ω via $A/T \cong \mathbb{Z}^2$. The second system $(\Sigma, \mathcal{M}, \nu, \mathbb{Z}^2)$ is defined as follows: the elements of Σ are all the labellings of the chambers of $\mathcal{A}$ by elements of $\mathbb{Z}/2\mathbb{Z}$ so that for any chamber the sum of the labels of the three chambers adjacent to it is 0. See Figure 3.2:

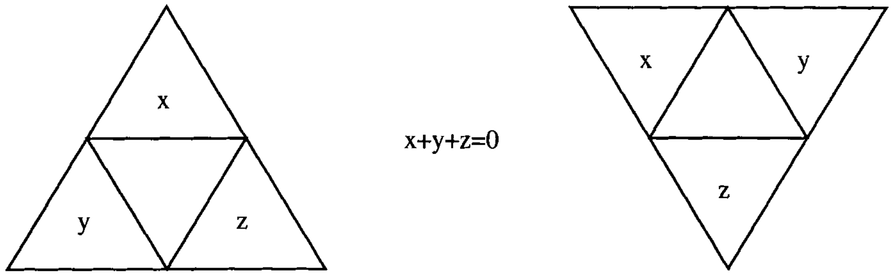

Figure 3.2

Σ is a compact abelian group, let ν be the normalized Haar measure on it. Again $\mathbb{Z}^2$ acts by translations of the apartment $\mathcal{A}$ via $A/T \cong \mathbb{Z}^2$. This system is a variant of the Ledrappier system. It is 2-mixing but not 3-mixing (see [Le]). By results of B. Kitchens and K. Schmidt, and of D. Lind (see [K-S1] Theorem 1.8, Proposition 2.12, Proposition 4.6), for every nonzero $(n,m) \in \mathbb{Z}^2$ the corresponding one dimensional system $(\Sigma, \mathcal{M}, \nu, T_{(n,m)})$ is Bernoulli and its entropy is the same as that of the corresponding one dimensional system $(\Omega, \mathcal{F}, \mu, S_{(n,m)})$. (Kitchens and Schmidt prove that the directional entropy function of this (and many other) system is piecewise linear. One can check that the "linearity cones" are the same and that the entropy coincides for translations by elements of $\mathbb{Z}^2$ which form bases for these cones.) Thus for any $(n,m) \in \mathbb{Z}^2$ the corresponding one dimensional systems are isomorphic. On the other hand, in [Moz1] it is shown that the system $(\mathrm{PGL}(3,\mathbb{Q}_2), \mathcal{B}, \mu, G)$, and hence also $(\Omega, \mathcal{F}, \mu, A)$, is mixing of all orders. Hence the systems are not isomorphic as 2-dimensional systems. ∎

A QUATERNION LATTICE IN $G = \mathrm{PGL}(2,\mathbb{Q}_p) \times \mathrm{PGL}(2,\mathbb{Q}_l)$. In this section we examine a specific example and describe explicitly the corresponding subshift of finite type. Let $G = \mathrm{PGL}(2,\mathbb{Q}_p) \times \mathrm{PGL}(2,\mathbb{Q}_l)$ where $p, l \equiv 1 \pmod 4$ are two distinct primes . The building Δ associated with G is the product of the buildings Δ_p and Δ_l associated with $\mathrm{PGL}(2,\mathbb{Q}_p)$ and $\mathrm{PGL}(2,\mathbb{Q}_l)$, i.e., it is the product of two regular trees of degree $p+1$ and $l+1$ respectively. The cells of Δ are squares corresponding to products of two edges. An apartment in Δ is the product of apartments in Δ_p and Δ_l, i.e., a plane tesselated by squares. Let $O_p \in \Delta_p$ (resp. $O_l \in \Delta_l$) be the vertex stabilized by $\mathrm{PGL}(2,\mathbb{Z}_p)$ (resp. $\mathrm{PGL}(2,\mathbb{Z}_l)$). Fix a Cartan subgroup of G:

$$A = \left\{ \left(\begin{pmatrix} \lambda & \\ & \lambda^{-1} \end{pmatrix}, \begin{pmatrix} u & \\ & u^{-1} \end{pmatrix} \right) \mid \lambda \in \mathbb{Q}_p^*,\ u \in \mathbb{Q}_l^* \right\}$$

The stabilizer of the apartment $\mathcal{A}$ associated with A is:

$$T = \left\{ \left(\begin{pmatrix} \lambda & \\ & \lambda^{-1} \end{pmatrix}, \begin{pmatrix} u & \\ & u^{-1} \end{pmatrix} \right) \mid \lambda, \lambda^{-1} \in \mathbb{Z}_p^*,\ u, u^{-1} \in \mathbb{Z}_l^* \right\}.$$

Notice that G does not preserve the labelling of the building. Let $\mathbb{H}$ be the rational quaternions, $\mathbb{H}(\mathbb{Z})$ the integer quaternions. Let $\psi\colon \mathbb{H} \to G$ be the embedding:

$$\begin{aligned} &\psi(x_0 + x_1\mathrm{i} + x_2\mathrm{j} + x_3\mathrm{k}) \\ &= \left(\begin{pmatrix} x_0 + x_1\mathrm{i}_p & x_2 + x_3\mathrm{i}_p \\ -x_2 + x_3\mathrm{i}_p & x_0 - x_1\mathrm{i}_p \end{pmatrix}, \begin{pmatrix} x_0 + x_1\mathrm{i}_l & x_2 + x_3\mathrm{i}_l \\ -x_2 + x_3\mathrm{i}_l & x_0 - x_1\mathrm{i}_l \end{pmatrix} \right) \end{aligned}$$

where $\mathrm{i}_p \in \mathbb{Q}_p$ and $\mathrm{i}_l \in \mathbb{Q}_l$ satisfy ${\mathrm{i}_p}^2 = -1$, ${\mathrm{i}_l}^2 = -1$.

Define $\tilde{\Gamma} = \{x = x_0 + x_1\mathrm{i} + x_2\mathrm{j} + x_3\mathrm{k} \in \mathbb{H}(\mathbb{Z}) \mid x \equiv 1(\mathrm{mod}\ 2)\ |x|^2 = p^r l^s\}$. It follows that $\Gamma = \psi(\tilde{\Gamma})$ is a uniform lattice in G (see [Tam]).

Proposition 3.6: *Γ is torsion free.*

Proof: Let $x = x_0 + x_1\mathrm{i} + x_2\mathrm{j} + x_3\mathrm{k} \in \tilde{\Gamma} \setminus \mathbb{Q}$. We have to show that no (nonzero) power of the matrix

$$M = \begin{pmatrix} x_0 + x_1 i & x_2 + x_3 i \\ -x_2 + x_3 i & x_0 - x_1 i \end{pmatrix} \in \mathrm{GL}_2(\mathbb{C})$$

is a scalar matrix. The eigenvalues of M are the roots u_1, u_2 of the polynomial $z^2 - 2x_0 z + p^r l^s$ where $|x|^2 = p^r l^s$. We have to verify that for $d \in \mathbb{N}$, $u_1^d \neq u_2^d$. Recall that $p, l \equiv 1(\mathrm{mod}\ 4)$, $x_0 \equiv 1(\mathrm{mod}\ 2)$. Denote:

$$x_0 = 1 + 2b, \quad p^r l^s = 1 + 4k, \quad c = k - b - b^2$$

Then

$$u_1 = x_0 + 2\sqrt{c}i, \quad u_2 = x_0 - 2\sqrt{c}i, \quad c \in \mathbb{N}.$$

If $u_1^d = u_2^d$ then u_1/u_2 is, w.l.o.g., a primitive root of unity of order d. Since u_1/u_2 belongs to a quadratic extension of $\mathbb{Q}$, $d = 1, 2, 3, 4$ or 6.

(1) $d \neq 1$ because $c > 0$.

(2) $d \neq 2$ because $x_0 \neq 0$.

(3) $d \neq 3$. We have

$$u_1^3 = x_0(x_0^2 - 12c) + (6a_0^2 - 8c)\sqrt{c}i,$$

$$u_1^3 = x_0(x_0^2 - 12c) - (6a_0^2 - 8c)\sqrt{c}i.$$

Since x_0 is odd $6x_0^2 - 8c \neq 0$, hence $u_1^3 \neq u_2^3$.

(4) $d \neq 4$. Otherwise $u_1/u_2 = \pm i$, hence $x_0 = \pm 2\sqrt{c}$ which is impossible since $2\sqrt{c}$ is either even or irrational whereas x_0 is odd.

(5) $d \neq 6$. Assume that u_1/u_2 is a root of unity of order 6 (exactly). Since $u_1 = \bar{u}_2$ it follows that the angle between u_1 and the real axis is $\pi/6$. W.l.o.g. $x_0 > 0$.

$$\frac{2\sqrt{c}}{x_0} = \tan \pi/6 = 1/\sqrt{2} \implies 2\sqrt{2c} = x_0.$$

This is impossible since $2\sqrt{2c}$ is either even or irrational whereas x_0 is odd.

It follows that no power of M is scalar and hence Γ is torsion free. ∎

Define

$$\tilde{\Gamma}_p = \left\{x \in \tilde{\Gamma} \mid |x|^2 = p^r,\ r \in \mathbb{N}\right\}, \quad \Gamma_p = \psi(\tilde{\Gamma}_p),$$
$$\tilde{\Gamma}_l = \left\{x \in \tilde{\Gamma} \mid |x|^2 = l^s,\ s \in \mathbb{N}\right\}, \quad \Gamma_l = \psi(\tilde{\Gamma}_l).$$

Notice that

$$\Gamma_p \subset \mathrm{PGL}(2, \mathbb{Q}_p) \times \mathrm{PGL}(2, \mathbb{Z}_l) \quad \Gamma_l \subset \mathrm{PGL}(2, \mathbb{Z}_p) \times \mathrm{PGL}(2, \mathbb{Q}_l).$$

THEOREM (see [Lu] Lemma 7.4.1): *The projection of* Γ_p *in* $\mathrm{PGL}(2, \mathbb{Q}_p)$ *is a lattice acting transitively on the vertices of the tree* Δ_p *associated with* $\mathrm{PGL}(2, \mathbb{Q}_p)$. *Similarly for* Γ_l, $\mathrm{PGL}(2, \mathbb{Q}_l)$.

PROPOSITION 3.7: Γ *acts freely transitively on the vertices of* Δ.

Proof: Since Γ is torsion free it follows that the action is free (a stabilizer is a compact subgroup which must be finite because Γ is discrete). Denote $O = (O_p, O_l) \in \Delta$. Let $(x, y) \in \Delta$ be any vertex. We will show that there exists $\gamma \in \Gamma$ of the form $\gamma = \gamma_l \gamma_p$, $\gamma_p \in \Gamma_p$, $\gamma_l \in \Gamma_l$ so that $\gamma O = (\gamma O_p, \gamma O_l) = (x, y)$. ($G$ acts on each of the trees Δ_p, Δ_l via its projection on the corresponding group.) Since

Γ_l acts (freely) transitively on the vertices of Δ_l there exists (unique) $\gamma_l \in \Gamma_l$ so that $\gamma_l O_l = y$. Let $x' = \gamma_l^{-1}x \in \Delta_p$. There exists (unique) $\gamma_p \in \Gamma_p$ so that $\gamma_p O_p = x'$. Let $\gamma = \gamma_l\gamma_p$. It satisfies:

$$\gamma O = (\gamma_l\gamma_p O_p, \gamma_l\gamma_p O_l) = (\gamma_l x', \gamma_l O_l) = (x, y).$$

Notice that in the same way it follows that there exists $\gamma' \in \Gamma_p\Gamma_l$ so that $\gamma'(O_p, O_l) = (x, y)$. ∎

COROLLARY 3.8: *$\Gamma = \Gamma_p\Gamma_l = \Gamma_l\Gamma_p$ and the decomposition is unique.*

Proof: Since Γ acts freely on the vertices of the building it follows that if $\gamma O = \gamma' O$ then $\gamma = \gamma'$. The proof of Proposition 3.7 shows that for any $\gamma \in \Gamma$ there are $\gamma' \in \Gamma_p\Gamma_l$ and $\gamma'' \in \Gamma_l\Gamma_p$ such that $\gamma O = \gamma' O = \gamma'' O$; hence $\Gamma = \Gamma_p\Gamma_l = \Gamma_l\Gamma_p$. To see the uniqueness of the decomposition let $\gamma_p, {\gamma'}_p \in \Gamma_p$, $\gamma_l, {\gamma'}_l \in \Gamma_l$ so that $\gamma = \gamma_p\gamma_l = {\gamma'}_p{\gamma'}_l$. Examine the action of γ on $O_p \in \Delta_p$. $\gamma O_p = \gamma_p\gamma_l O_p = \gamma_p O_p$ as well as $\Gamma O_p = {\gamma'}_p{\gamma'}_l O_p = {\gamma'}_p O_p$. Since Γ_p acts freely on Δ_p it follows that $\gamma_p = {\gamma'}_p$ hence also $\gamma_l = {\gamma'}_l$. The uniqueness of the factorization $\Gamma = \Gamma_l\Gamma_p$ is shown in the same way. ∎

Let

$$\underline{\tilde{A}} = \left\{a \in \tilde{\Gamma} \mid |a|^2 = p\right\}, \quad \underline{A} = \psi(\underline{\tilde{A}}),$$
$$\underline{\tilde{B}} = \left\{a \in \tilde{\Gamma} \mid |a|^2 = p\right\}, \quad \underline{B} = \psi(\underline{\tilde{B}}).$$

THEOREM (see [Lu]): *$|\underline{A}| = p + 1$. $\Gamma_p =< \underline{A} >$ is a free group with $(p+1)/2$ generators. $\underline{A}$ is a symmetric set of generators. $|\underline{B}| = l + 1$. $\Gamma_l =< \underline{B} >$ is a free group with $(l+1)/2$ generators. $\underline{B}$ is a symmetric set of generators.*

An element $a \in \underline{A}$ is the image of a matrix with determinant p, hence it maps the vertex $O_p \in \Delta_p$ to one of its neighbours (see [Ser]). Since Γ_p acts freely it follows that the $p+1$ elements of $\underline{A}$ maps O_p to the $p+1$ vertices adjacent to O_p. Similar assertions hold for Γ_l. Recall that $\Gamma_p \subset \mathrm{stab}O_l$, $\Gamma_l \subset \mathrm{stab}O_p$. We conclude that:

PROPOSITION 3.9:

(1) *The elements of $\underline{A} \subset \Gamma$ map the vertex $O = (O_p, O_l) \in \Delta$ to the $p+1$ adjacent vertices (x, O_l) ($x \in \Delta_p$ adjacent to O_p).*
(2) *The elements of $\underline{B} \subset \Gamma$ map the vertex $O = (O_p, O_l) \in \Delta$ to the $l+1$ adjacent vertices (O_p, y) ($y \in \Delta_l$ adjacent to O_l).*

Note that the above vertices $\{(x, O_l), (O_p, y)\}$ are exactly all the vertices adjacent to $O = (O_p, O_l)$ in the 1-skeleton of Δ.

PROPOSITION 3.10: *The 1-skeleton of Δ can be identified with the (right) Cayley graph of Γ with respect to the generators $\underline{A} \cup \underline{B}$.*

Proof: We have already shown that $\gamma \leftrightarrow \gamma O$ is a one to one correspondence between the elements of Γ and the vertices of the building Δ. Since the action of Γ on Δ preserve adjacency relations, it follows that the vertices corresponding to $\gamma, \gamma' \in \Gamma$ are adjacent if and only if the vertices O and $\gamma^{-1}\gamma' O$ are adjacent, which is equivalent by Proposition 3.9 to $\gamma^{-1}\gamma' \in \underline{A} \cup \underline{B}$. Hence this correspondence identifies the 1-skeleton of Δ with the right Cayley graph of Γ. ∎

We obtain a labelling of the oriented edges of Δ by the elements of $\underline{A} \cup \underline{B}$. Note that if we follow some closed path along the edges and multiply the labels, we obtain the identity. Hence:

COROLLARY 3.11: $\underline{A}\,\underline{B} = \underline{B}\,\underline{A}$ *(in a unique way).*

This together with the fact that Γ acts freely transitively on the vertices and preserves the labelling of the edges by $\underline{A} \cup \underline{B}$ give the following description of the quotient complex $\Gamma \backslash \Delta$.

THEOREM 3.2: *The complex $Y = \Gamma \backslash \Delta$ has a single vertex, $(p+1)/2 + (l+1)/2$ edges and $(p+1)(l+1)$ faces (squares). $(p+1)/2$ of the edges are labelled, after being oriented, by the elements of $\underline{A}$ s.t. the opposite orientations are labeled by an element and its inverse. Similarly, the other $(l+1)/2$ edges are labelled by the elements of $\underline{B}$. For any $a \in \underline{A}$, $b \in \underline{B}$ there exists a unique pair $a' \in \underline{A}$, $b' \in \underline{B}$ so that $ab = b'a'$. Corresponding to them is a square whose edges are labeled as in Figure 3.3.*

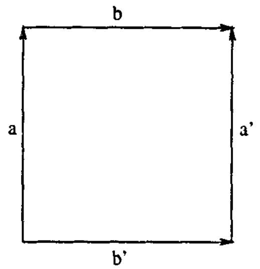

Figure 3.3

Each such square is glued to the 1-skeleton so that its vertices are identified with the single vertex and the edges are glued according to the coloring (together with orientation).

This finite complex defines a two-dimensional subshift of finite type Ω. Call an edge of Y which is labelled by an element of $\underline{A}$ (resp. $\underline{B}$) vertical (resp. horizontal). An element $f \in \Omega$ may be identified with a labelling of the (oriented) edges of an apartment $\mathcal{A}$ (which is a plane tesselated by squares) by the elements of $\underline{A} \cup \underline{B}$ (we always label the two orientations of an edge by reciprocal elements) so that:

(1) Vertical edges are labelled by elements of $\underline{A}$.
(2) Horizontal edges are labelled by elements of $\underline{B}$.
(3) Two consecutive edges are not labelled by reciprocal elements.
(4) If we look at any four edges of a square, oriented counter-clockwise, and multiply their labels cyclically, then the product is the identity.

The group $\mathbb{Z}^2 \cong A/T$ acts on Ω. Denote the action by $S_v \colon \Omega \to \Omega$ for $v \in \mathbb{Z}^2$. The action of $A/T \cong \mathbb{Z}^2$ on $\mathcal{A}$ enables us to associate with every element of $\mathbb{Z}^2$ a vertex in $\mathcal{A}$ (after a choice of an origin). For $\omega \in \Omega$, define the "stabilizer lattice" $\mathcal{L}_\omega = \{v \in \mathbb{Z}^2 \mid S_v\omega = \omega\}$. For $\omega \in \Omega$ and $u, v \in \mathbb{Z}^2$ let $\omega(u,v) \in \Gamma$ be the product of the elements of Γ written along any path of edges from the vertex corresponding to u to the vertex corresponding to v (notice that $\omega(u,v)$ is independent of the path). For a periodic point $\omega \in \Omega$ let $G(\omega) = \{\omega(u,v) \mid u, v \in \mathcal{L}_\omega\}$.

PROPOSITION 3.12: $\mathrm{rank}\mathcal{L}_\omega = 0, 2$; *i.e., if $\omega \in \Omega$ has a period in some direction, it has a period also in another direction (and hence a finite $\mathbb{Z}^2$ orbit).*

Proof: Assume $\mathrm{rank}\,\mathcal{L}_\omega > 0$. If there is some $v = (s,r) \in \mathcal{L}_\omega$ such that $|r|, |s| > 0$ then it follows, see Proposition 2.14, that ω is a periodic point and $\mathrm{rank}\,\mathcal{L}_\omega = 2$. Assume that $s = 0$, $r > 0$ (the case $s \neq 0$, $r = 0$ is dealt with in a similar way). This implies that ω is determined by its restriction to a fundamental domain for the action of $\mathbb{Z}v$ on $\mathcal{A}$, which we can take to be a horizontal strip of height r. The labelling of the upper and lower boundary edges of the strip is identical. Denote this sequence of labels by $(b_i)_{i\in\mathbb{Z}}$. For each vertical section of the strip of height r multiply the labels and denote the sequence of elements obtained by $(\alpha_i)_{i\in\mathbb{Z}}$. See Figure 3.4.

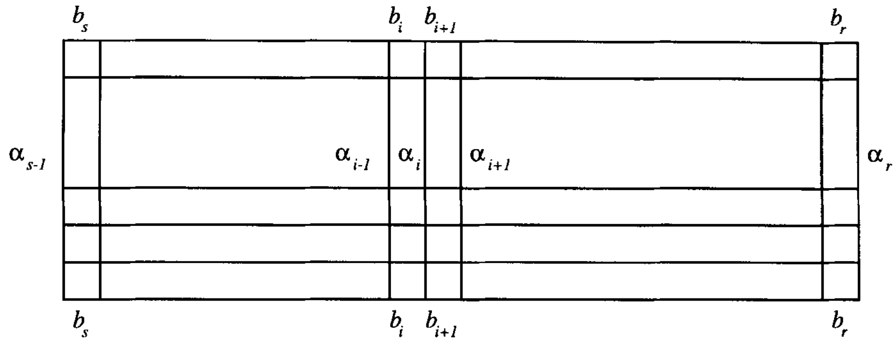

Figure 3.4

Let $\beta_i^j = \prod_{k=i}^j b_k \in \Gamma_l$. It follows from the definition of the subshift of finite type Ω that

$$\alpha_i\beta_{i+1}^j = \beta_{i+1}^j\alpha_j, \quad \alpha_i^{-1}(\beta_m^i)^{-1} = (\beta_m^i)^{-1}\alpha_{m-1}^{-1}.$$

Apply these to the vertex $O_l \in \Delta_l$:

$$\alpha_i\beta_{i+1}^j O_l = \beta_{i+1}^j\alpha_j O_l = \beta_{i+1}^j O_l,$$

$$\alpha_i^{-1}(\beta_m^i)^{-1}O_l = (\beta_m^i)^{-1}\alpha_{m-1}^{-1}O_l = (\beta_m^i)^{-1}O_l.$$

This means that for any $i \in \mathbb{Z}$ the sequence of vertices $(\cdots(\beta_{i-1}^i)^{-1}O_\ell, (\beta_{i-1}^i)^{-1}O_\ell, (\beta_i^i)^{-1}O_\ell, O_\ell, \beta_{i+1}^{i+1}O_\ell, \beta_{i+1}^{i+2}O_\ell, \cdots)$ are all fixed by α_i. These vertices lie along an infinite line in the tree Δ_l. A nontrivial element $\alpha_i \in \Gamma_p$ has at most one fixed line in the tree Δ_l. It follows that the vertices $((\beta_m^i)^{-1}O_l)\,(m \leq i, O_l, \beta_{i+1}^j O_l)\,(j > i)$ are determined by α_i together with a choice of orientation of its fixed line. This implies that the pair (α_i, b_{i+1}) determines the labelling of the whole horizontal strip. Since the number of such pairs is finite, some pairs repeat along the strip and it follows that the labelling of the strip is periodic. Hence rank $\mathcal{L}_\omega = 2$. ∎

PROPOSITION 3.13: *Let $\omega \in \Omega$ be a periodic point. Then*

(1) *$G(\omega)$ is a free abelian group of rank 2.*
(2) *$G(\omega)$ determines ω up to (4 fold) symmetry.*
(3) *$G(\omega)$ is determined by any $e \neq \gamma \in G(\omega)$.*

Proof:

(1) For $u, v \in \mathcal{L}_\omega$, $\omega(u,v) = \omega(v,u)^{-1}$. For $u, v, w, t \in \mathcal{L}_\omega$ the periodicity of ω implies that $v + t - w \in \mathcal{L}_\omega$ and $\omega(w,t) = \omega(v, v+t-w)$. Hence

$$\omega(u,v)\omega(w,t) = \omega(u,v)\omega(v,v+t-w) = \omega(u,v+t-w) \in \Gamma(\omega).$$

Hence $G(\omega)$ is a group. The above also shows that the map $\chi: \mathcal{L}_\omega \to G(\omega)$ defined by $\chi(u) = \omega((0,0),u)$ is an epimorphism. Notice that if we lift the map $\omega: \mathcal{A} \to Y$ to a map $\tilde{\omega}: \mathcal{A} \to \Delta$ so that the vertex corresponding to $(0,0)$ is mapped to the vertex $O = (O_p, O_l)$ (which corresponds to $e \in \Gamma$), then for any $u \in \mathcal{L}_\omega$ the vertex u is mapped to $\omega((0,0),u)O = \chi(u)O$. It follows from the fact that $\tilde{\omega}$ is an embedding of $\mathcal{A}$ in Δ that χ is injective. Hence $G(\omega)$ is isomorphic to $\mathcal{L}_\omega$ a free abelian group of rank 2.

(2) Choose $\gamma \in G(\omega) \setminus (\Gamma_p \cup \Gamma_l)$ (it exists since $G(\omega)$ is a free abelian group of rank 2). Let $\gamma = \alpha\beta$ where $\alpha \in \Gamma_p$, $\beta \in \Gamma_l$ are both nontrivial. Denote by $r > 0$ the length of α as a word in the generators $\underline{A}$ of the free group Γ_p. Denote by $s > 0$ the length of β as a word in the generators $\underline{B}$ of the free group Γ_l. It follows that γ equals one of the following four possibilities: $\omega((0,0),(\pm r, \pm s))$. Each of these determines the sequence of labels written along a zigzagging line in the apartment which determines the whole element ω. We conclude that there are four possible elements of Ω corresponding to the same group $G(\omega)$. These elements are related to one another by reflecting along the horizontal and vertical axis through the origin of $\mathcal{A}$.

(3) The above shows that an element of $G(\omega) \setminus (\Gamma_p \cup \Gamma_l)$ determines the point ω (up to symmetry) and hence the group $G(\omega)$. We have to deal separately with the case of nontrivial $\gamma \in G(\omega) \cap (\Gamma_p \cup \Gamma_l)$. Assume, for example, $e \neq \gamma \in G(\omega) \cap \Gamma_p$. Assume that there, for some periodic $\omega' \in \Omega$, $\gamma \in G(\omega')$. Notice that both $G(\omega) \cap \Gamma_l \neq \{e\}$ and $G(\omega') \cap \Gamma_l \neq \{e\}$. Also $G(\omega) \cap \Gamma_l, G(\omega') \cap \Gamma_l \subset \mathrm{Centralizer}_{\Gamma_l}(\gamma)$. By considering the centralizer of the quaternion corresponding to γ in $\tilde{\Gamma}_l$ we see, using the fact that this quaternion is not in $\mathbb{Q}$, that $\mathrm{Centralizer}_{\Gamma_l}(\gamma)$ is a (nontrivial) commutative subgroup of the free group Γ_l; hence it is a cyclic subgroup of Γ_l. It follows that there exists some nontrivial element $\gamma' \in G(\omega) \cap G(\omega') \cap \Gamma_l$. Hence we obtain an element $\gamma\gamma' \in (G(\omega) \cap G(\omega')) \setminus (\Gamma_p \cup \Gamma_l)$ which, by the above, implies that $G(\omega) = G(\omega')$. ∎

It will be convenient also to think about the symbols written on the edges as integral quaternions rather than elements of $\Gamma \subset \mathrm{PGL}(2,\mathbb{Q}_p) \times \mathrm{PGL}(2,\mathbb{Q}_l)$. We will abuse notation and refer to elements both as quaternions and as elements of G (we will take an integral quaternion with relatively prime coefficients representing the element). Let $\tau\colon \mathbb{H}(\mathbb{Q}) \setminus \mathbb{Q} \to \mathbb{P}^2(\mathbb{Q})$ be defined by: $\tau(x_0 + x_1\mathrm{i} + x_2\mathrm{j} + x_3\mathrm{k}) = (x_1\colon x_2\colon x_3)$. Two quaternions $x, y \in \mathbb{H}(\mathbb{Q}) \setminus \mathbb{Q}$ commute if and only if $\tau(x) = \tau(y)$.

Proposition 3.14: *Let ω be a periodic point and $G(\omega)$ be the corresponding group. Then $G(\omega)$ is determined by the point $\tau(\gamma) = (c_1\colon c_2\colon c_3) \in \mathbb{P}^2(\mathbb{Q})$ corresponding to a quaternion representing any nontrivial element $\gamma \in G(\omega)$.*

Proof: Let $\omega, \omega' \in \Omega$ be two periodic points so that there are nontrivial elements $\gamma \in G(\omega)$, $\gamma' \in G(\omega')$ so that $\tau(\gamma) = \tau(\gamma')$. We may assume without loss of generality that $\gamma, \gamma' \in \Gamma_l$ (clearly the image under τ of any nontrivial element in an abelian group doesn't depend on the particular element). We see that $<\gamma>$ and $<\gamma'>$ are cyclic subgroups of a free group which commute with each other. Hence they have a nontrivial intersection, i.e., we obtain some $e \neq \gamma'' \in G(\omega) \cap G(\omega')$; hence it follows that $G(\omega) = G(\omega')$. ∎

Let $\bar{c} = (c_1 : c_2 : c_3) \in \mathbb{P}^2(\mathbb{Q})$. Associate with it a quadratic form $Q_{\bar{c}}(x,y)$ as follows: we may assume that $c_1, c_2, c_3 \in \mathbb{Z}$ and are relatively prime. Let $n = c_1^2 + c_2^2 + c_3^2$ and define $Q_{\bar{c}}(x,y) = x^2 + 4ny^2$. We will say that the form $Q_{\bar{c}}$ represents $p^r l^s$ if there are $x, y \in \mathbb{Z}$ relatively prime to each other and to pl so that $p^r l^s = Q_{\bar{c}}(x,y)$. (Notice that this implies that also $\gcd(n, pl) = 1$.)

Proposition 3.15: *$\bar{c} = (c_1 : c_2 : c_3) \in \mathbb{P}^2(\mathbb{Q})$ corresponds to a periodic point $\omega \in \Omega$ if and only if $Q_{\bar{c}}$ represents $p^r l^s$ for some $r, s \geq 1$.*

Proof: Let $\omega \in \Omega$ be a periodic point and $\bar{c} \in \mathbb{P}^2(\mathbb{Q})$ be the corresponding point. Let $\bar{c} = (c_1 : c_2 : c_3)$ with $c_1, c_2, c_3 \in \mathbb{Z}$ relatively prime, $n = c_1^2 + c_2^2 + c_3^2$. Choose a quaternion $z = z_0 + z_1\mathrm{i} + z_2\mathrm{j} + z_3\mathrm{k} = \omega((0,0),(r,s)) \in G(\omega)$ for some $r, s \in \mathbb{N}$, $(r,s) \in \mathcal{L}_\omega$. $z \equiv 1 \pmod 2$, $(z_1\colon z_2\colon z_3) = \bar{c}$, $|z|^2 = p^r l^s$. Define $x_0 = z_0$, $y_0 = z_1/2c_1 = z_2/2c_2 = z_3/2c_3$. It follows that $p^r l^s = x_0^2 + 4ny_0^2 = Q_{\bar{c}}(x_0, y_0)$. We have to show that $\gcd(x_0 y_0, pl) = 1$. Consider $z^2 = (x_0^2 - 4ny_0^2) + 2x_0 z_1 i + 2x_0 z_2 j + 2x_0 z_3 k \in G(\omega)$. The corresponding vertex $z^2 O \in \Delta$ is at a distance of $2s$ horizontal edges and $2r$ vertical edges (this follows from considering the lifting of ω to a map of $\mathcal{A}$ to the building Δ). $|z^2|^2 = p^{2r} l^{2s}$; hence

$\gcd(x_0^2 - 4ny_0^2, 2x_0z_1, 2x_0z_2, 2x_0z_3) = 1$, otherwise they would be divisible by p or l and the quaternion z^2 would have been equivalent to a quaternion of smaller absolute value, which would have implied that the vertex z^2O is closer to O. Hence $\gcd(x_0, pl) = 1$; otherwise if, for example, $p|x_0$, then since $p|x_0^2 + 4ny_0^2 \implies p|4ny_0^2$ we conclude that p divides all the coefficients of z^2. Similarly $\gcd(y_0, pl) = 1$. If, for example, $p|y_0$ then also $p|x_0$.

Conversely, assume $\bar{c} = (c_1 : c_2 : c_3) \in \mathbb{P}^2(\mathbb{Q})$, $c_1, c_2, c_3 \in \mathbb{Z}$ relatively prime, $n = c_1^2 + c_2^2 + c_3^2$ and there exists a representation $p^r l^s = Q_{\bar{c}}(x_0, y_0) = x_0^2 + 4ny_0^2$ for some $r, s \in \mathbb{N}$ with $\gcd(x_0y_0, pl) = 1$. Define a quaternion $z = x_0 + 2y_0c_1i + 2y_0c_2j + 2y_0c_3k$. Notice that $z \equiv 1 (\mathrm{mod}\ 2)$, $|z|^2 = p^r l^s$ and $z_0 + z_1\mathrm{i} + z_2\mathrm{j} + z_3\mathrm{k} \in \Gamma$ (we abuse notation and identify $\tilde{\Gamma}$ and Γ). We can decompose $z = \alpha\beta$ where $\alpha \in \Gamma_p$, $\beta \in \Gamma_l$. Define a tiling ω as follows. Write α and β periodically along a zigzagging line in $\mathcal{A}$ as in Figure 3.5.

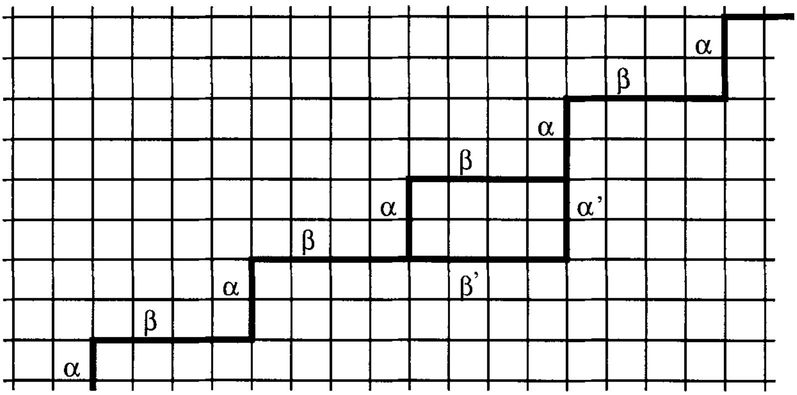

Figure 3.5

Let $\alpha' \in \Gamma_p$, $\beta' \in \Gamma_l$ be so that $\alpha\beta = \beta'\alpha'$. In order that this labelling be part of a legal element of Ω, it is necessary and sufficient that both $\alpha'\alpha$ and $\beta\beta'$ will be reduced words in the corresponding free groups Γ_p and Γ_l. The word $\alpha'\alpha$ is not reduced if α begins by a quaternion $a \in \underline{A}$ and α' ends with the quaternion $\bar{a} \in \underline{A}$. This implies that $p = \bar{a}a|\alpha'\alpha$. Similarly, if $\beta\beta'$ is not reduced then $l|\beta\beta'$. Thus if either of them is not reduced then p or l divides the quaternion $z^2 = \alpha\beta\beta'\alpha' = \beta'\alpha'\alpha\beta$. Notice that $z^2 = (x_0^2 - 4ny_0^2) + 4x_0y_0(c_1i + c_2j + c_3k)$. Hence if, for example, $p|z^2$, then $p|x_0^2 - 4ny_0^2$. Since also $p|p^r l^s = x_0^2 + 4ny_0^2$ it follows that $p|x_0$ contrary to the assumption. It follows that $\alpha'\alpha$ and $\beta\beta'$ are reduced and the above labelling can be completed to a legal labelling of the whole apartment giving $\omega \in \Omega$. Moreover, since the original zigzagging labelling was periodic, ω is a periodic point. Clearly $z \in G(\omega)$, hence $\bar{c} = \tau(z)$ corresponds to a periodic point. ∎

We remark that it follows from Propositions 3.13, 3.14, 3.15 and their proofs that $G(\omega)$ is its own centralizer in Γ. Starting with a periodic point $\omega \in \Omega$ we obtained a quadratic form which we will denote by Q_ω. We will see next that the quadratic form determines the "period", $\mathcal{L}_\omega$, of ω.

PROPOSITION 3.16: *Let $\omega \in \Omega$ be a periodic point, $Q_\omega(x,y) = x^2 + 4ny^2$ the quadratic form corresponding to it. Let $K = \mathbb{Q}(\sqrt{-n})$, $\mathcal{O} = \mathbb{Z} + 2\sqrt{-n}\mathbb{Z}$ an order in K. Then p, l split in $\mathcal{O}$, i.e., $p\mathcal{O} = P_1P_2$, $l\mathcal{O} = L_1L_2$ where P_1, P_2, L_1, L_2 are ideals in $\mathcal{O}$. These ideals may be viewed as elements of the class group, $Cl(\mathcal{O})$, of the ideals of norm relatively prime to the index, $2f$, of $\mathcal{O}$ in $\mathcal{O}_K$, the maximal order of K. $n = df^2$ where d is squarefree. Without loss of generality (up to exchanging P_1 with P_2 and/or L_1 with L_2), $\mathcal{L}_\omega = \{(s,r) \in \mathbb{Z}^2 \mid P_1^r L_1^s = \mathrm{id}\}$.*

Proof: There exists a basis $\{(s_1,r_1),(s_2,r_2)\}$ for $\mathcal{L}_\omega$ such that $r_1, s_1, r_2, s_2 \in \mathbb{N}$. Let $z = z_0 + z_1\mathrm{i} + z_2\mathrm{j} + z_3\mathrm{k} = \omega((0,0),(s_1,r_1))$, $y = z_1/2c_1 = z_2/2c_2 = z_3/2c_3$, $x = z_0$. $p^{r_1}l^{s_1} = x^2 + 4ny^2$ and $\gcd(x, pl) = 1$. Hence the ideals $p\mathcal{O}, l\mathcal{O}$ split in $\mathcal{O}$. Notice that $\gcd(n, pl) = 1$. $u = x + 2y\sqrt{-n} \in \mathcal{O}$ generated an ideal $u\mathcal{O}$ with norm $p^{r_1}l^{s_1} = x^2 + 4ny^2$ which is relatively prime to the index $2f$ of $\mathcal{O}$ in $\mathcal{O}_K$. It follows from the unique factorization of such ideals in $\mathcal{O}$ that w.l.o.g. $u\mathcal{O} = P_1^{r_1}L_1^{s_1}$ (notice that $u\mathcal{O} \not\subset p\mathcal{O}$, hence in the factorization of $u\mathcal{O}$ only one of P_1 or P_2 appears; similarly only one of L_1 or L_2 appears). (For properties of factorization of ideals and class groups see [C], [La].) It follows that $P_1^{r_1}L_1^{s_1} = \mathrm{id}$ in $Cl(\mathcal{O})$. The same considerations applied to (s_2, r_2) give, via the quaternion $z' = \omega((0,0),(s_2,r_2))$, an element $u' \in \mathcal{O}$ so that $u'\mathcal{O} = P_i^{r_2}L_j^{s_2}$. We have to show that $i = j = 1$.

$$u = x + 2y\sqrt{-n}, \quad z = x + 2y(c_1 i + c_2 j + c_3 k),$$
$$u' = x' + 2y'\sqrt{-n}, \quad z' = x' + 2y'(c_1 i + c_2 j + c_3 k).$$

Examine the products:

$$zz' = xx' - 4yy'n + 2(xy' + x'y)(c_1 i + c_2 j + c_3 k),$$
$$uu' = xx' - 4yy'n + 2(xy' + x'y)\sqrt{-n}.$$

Since $r_1, s_1, r_2, s_2 \geq 1$,

$$zz' = \omega((0,0),(s_1 + s_2, r_1 + r_2)).$$

Hence $\gcd(xx' - 4yy'n, pl) = 1$. It follows that the ideal $uu'\mathcal{O}$ is not divisible by $p\mathcal{O}$ or $l\mathcal{O}$. Since $uu'\mathcal{O} = u\mathcal{O}u'\mathcal{O} = P_1^{r_1}L_1^{s_1}P_i^{r_2}L_j^{s_2}$ it follows that $i = j = 1$. Since $\mathcal{L}_\omega = \langle(s_1, r_1), (s_2, r_2)\rangle$ it follows that $\mathcal{L}_\omega \subset \{(s,r) \in \mathbb{Z}^2 \mid P_1^r L_1^s = \mathrm{id}\}$. Assume that for some $(s,r) \in \mathbb{N}^2$, $P_1^r L_1^s = \mathrm{id}$ in $Cl(\mathcal{O})$. It follows that there exists $u = x + 2y\sqrt{-n} \in \mathcal{O}$ and $x^2 + 4ny^2 = p^r l^s$. Define $z = x + 2y(c_1 i + c_2 j + c_3 k)$. $|z|^2 = p^r l^s$, x is odd, hence $z \in \tilde{\Gamma}$. z commutes with $G(\omega)$; since $G(\omega)$ is its own centralizer it follows that $z \in G(\omega)$. Hence $(s,r) \in \mathcal{L}_\omega$, i.e., $\{(s,r) \in \mathbb{Z}^2 \mid P_1^r L_1^s = \mathrm{id}\} \cap \mathbb{N}^2 \subset \mathcal{L}_\omega$. It follows that $\{(s,r) \in \mathbb{Z}^2 \mid P_1^r L_1^s = \mathrm{id}\} \subset \mathcal{L}_\omega$. ∎

Let $\mathrm{r}_3(n)$ denote the number of representations of $n \in \mathbb{N}$ as the (ordered) sum of three relatively prime integers. For $(s,r) \in \mathbb{N}^2$ let

$$a_{(s,r)} = \#\{\omega \in \Omega \mid S_{(s,r)}\omega = \omega\}.$$

For $k \in \mathbb{N}$ let

$$b_k = \#\{\omega \in \Omega \mid \text{ the orbit of } \omega \text{ has size } k\}.$$

Proposition 3.17: *Given $n \in \mathbb{N}$ denote $\mathcal{O}_n = \mathbb{Z} + 2\sqrt{-n}\mathbb{Z}$. Then*

(1) $a_{(s,r)} = 4\sum_{n \in I_1(s,r)} \mathrm{r}_3(n) + 2\sum_{n \in I_2(s,r)} \mathrm{r}_3(n)$ *where*

$$I_1(s,r) = \{n \in \mathbb{N} \mid p\mathcal{O}_n = P_1P_2,\ l\mathcal{O}_n = L_1L_2,\ P_1^rL_1^s = \mathrm{id},\ P_1^rL_2^s = \mathrm{id}\},$$
$$I_2(s,r) = \{n \in \mathbb{N} \mid p\mathcal{O}_n = P_1P_2,\ l\mathcal{O}_n = L_1L_2,\ P_1^rL_1^s = \mathrm{id},\ P_1^rL_2^s \neq \mathrm{id} \text{ or } P_1^rL_1^s \neq \mathrm{id},\ P_1^rL_2^s = \mathrm{id}\}.$$

(2) $b_k = 4\sum_{n \in I(k)} \mathrm{r}_3(n)$ *where*

$$I(k) = \{n \in \mathbb{N} \mid p\mathcal{O}_n = P_1P_2,\ l\mathcal{O}_n = L_1L_2,\ \#\langle P_1, L_1\rangle = k\}.$$

Proof: We verify (2) first. The orbit of a periodic point $\omega \in \Omega$ has size k if and only if $\mathcal{L}_\omega$ is of index k in $\mathbb{Z}^2$ which, by Proposition 3.16, is equivalent to $\#\langle P_1, L_1\rangle = k$. Thus the points whose orbit size is k correspond to quadratic forms $x^2 + 4ny^2$ such that $n \in I(k)$. To each such quadratic form correspond $\mathrm{r}_3(n)$ points in $\mathbb{P}^2(\mathbb{Q})$. Each of these points corresponds to exactly four different points in Ω (see Proposition 3.13). Notice that the four points are obtained from one another by reflection at the horizontal and vertical axis through the origin, all have the same orbit size and are distinct (because two consecutive edges are never labeled by an element and its inverse (conjugate)). We turn to (1). By

Proposition 3.16, if $\omega \in \Omega$ satisfies $S_{(s,r)}\omega = \omega$ then, up to exchanging P_1 with P_2 and/or L_1 with L_2, $P_1^r L_1^s = \mathrm{id}$ in the corresponding class group. Conversely, the latter equality implies that either $S_{(s,r)}\omega = \omega$ or $S_{(s,-r)}\omega = \omega$, where both hold simultaneously exactly when the corresponding n belongs to $I_1(s,r)$. We conclude that each $\omega \in \Omega$ such that $S_{(s,r)}\omega = \omega$ corresponds to a quadratic form $x^2 + 4ny^2$ such that $n \in I_1(s,r) \cup I_2(s,r)$. To a quadratic form correspond four periodic points in Ω (see Proposition 3.13). Of these, 2 or 4 satisfy $S_{(s,r)}\omega = \omega$ according to whether $n \in I_2(s,r)$ or $n \in I_1(s,r)$. ∎

The subshift of finite type Ω may be viewed as a subset of $S^{\mathbb{Z}^2}$ where the set of symbols S is the collection of $(p+1)(l+1)$ unit squares whose edges are labeled by the elements of $\underline{A} \cup \underline{B}$ according to the rules defined above, and such that the adjacency rules are defined by two $0-1$ matrices: H defining the legal horizontal adjacencies, and V defining the legal vertical adjacencies. From the definition of Ω and Corollary 3.11 it follows that if two symbols $x, y \in S$ may be neighbours at positions (i,j) and $(i+1, j+1)$, then there exist unique possible symbols $e, f \in S$ at positions $(i+1, j), (i, j+1)$ respectively. This implies that the two matrices H, V commute. This implies that for $s, r \geq 1$, $a_{(s,r)} = \mathrm{tr} H^s V^r$. It follows that:

$$F_{(s,r)}(z) = \exp\Big(\sum_{n\geq 1} \frac{a_{(ns,nr)}}{n} z^n\Big) = \frac{1}{\det(I - zH^sV^r)}.$$

We end our discussion of the structure of the periodic points of this system by observing that there are in general several orbits of the same "shape":

Proposition 3.18: *To any $n \in N$ such that p, l split in $\mathcal{O}_n$ as $p\mathcal{O}_n = P_1P_2$, $l\mathcal{O}_n = L_1L_2$ there correspond $\mathrm{r}_3(n)/\#\langle P_1, L_1\rangle$ periodic orbits having the same periodic structure (i.e., the same lattice $\mathcal{L}_\omega \subset \mathbb{Z}^2$).*

Proof: Notice that all the points in a periodic orbit define the same quadratic form, i.e., if ω' is a translate of ω then $Q_\omega = Q_{\omega'}$. To see this, note that for $(s,r) \in \mathcal{L}_\omega = \mathcal{L}_{\omega'}$ the quaternions $x = \omega((0,0),(s,r))$ and $x' = \omega'((0,0),(s,r))$ are conjugate, hence have the same real part. Moreover, as we can conjugate one to the other by a rational quaternion having norm one and such that the denominators of its coefficients are divisible only by the primes p, l both of which are relatively prime to the coefficients of the quaternions x, x', it follows that the respective common divisors of the coefficients of the imaginary parts of x and of

x' are the same. This implies that $Q_\omega = Q_{\omega'}$. Given $n \in \mathbb{N}$ so that p, l split in $\mathcal{O}_n$ we fix some $s, r \in \mathbb{N}$ so that $P_1^r L_1^s = \mathrm{id}$. This means we can solve $Q_n(x, y) = p^r l^s$. Fix a solution (x_0, y_0). For any decomposition $n = c_1^2 + c_2^2 + c_3^2$ as the sum of three relatively prime squares we obtain a quaternion $z = x_0 + y_0(c_1 i + c_2 j + c_3 k)$. This quaternion determines a labelling of a $r \times s$ rectangle (where z is the product of the labels from the lower left corner to the upper right corner). Repeat this labelling along a bi-infinite sequence of rectangles having the upper right corner of one touching the lower left corner of another. This labelling may be completed in a unique way to a legal labelling of the whole plane giving an element $\omega \in \Omega$ which is periodic and whose quadratic form $Q_\omega = Q_n$. Thus we get $r_3(n)$ different periodic points of Ω all of which have the same periodic shape. ∎

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